Department of Physics and Astronomy

Heidelberg University

Master thesis

in Physics

submitted by

Zou, Kefei

born in Beijing, China

2022

$0\nu\beta\beta$ Decay in the Minimal Left-Right Symmetric Model

This Master thesis has been carried out by Zou, Kefei

at the

Max-Planck-Institut für Kernphysik

under the supervision of

Prof. Dr. Dr. h.c. Manfred Lindner

$0\nu\beta\beta$ Zerfall im minimalen links-rechts symmetrischen Modell:

In dieser Arbeit berechnen wir die differentielle Zerfallsrate von $0\nu\beta\beta$ im minimalen links-rechts-symmetrischen Modell (mLRSM) im Rahmen einer effektiven Feldtheorie bei niedrigen Energien. Wir studieren die grundlegende Modellbildung des mLRSM aus der Literatur und leiten die Massenskalen der beteiligten Teilchen ab. Diese Massen sowie die Kopplungen des Modells spielen eine entscheidende Rolle im $0\nu\beta\beta$-Prozess, um die Zerfallsamplituden zu berechnen. Wir geben die Amplitude jedes Diagramms an und berechnen die differentielle Zerfallsrate. Das Herleitungsverfahren vom Modell zur Zerfallsrate ist für Studenten hilfreich, um die Beziehung zwischen theoretischen Modellen und Experimenten zu verstehen und dient als Ausgangspunkt für weiterführende Forschung. Am Ende zeigen wir die resultierenden Ausdrücke der differentiellen Halbwertszeiten, Einzelelektron-Spektren sowie der Winkelkorrelationen, die aus den verschiedenen Beiträgen resultieren.

$0\nu\beta\beta$ decay in the minimal left-right symmetric model:

In this thesis we calculate the differential decay rate of $0\nu\beta\beta$ in the minimal left-right symmetric model (mLRSM) using the low energy effective approach. We study the fundamental model building of the mLRSM from the literature and derive the mass scales of the involving particles. These masses as well as the couplings of the model play a crucial role in the $0\nu\beta\beta$ process for amplitude derivations. We give the amplitude of each diagrams and calculate the differential decay rate. The derivation procedure from the model to the differential decay rate is helpful for students to understand the relation between theoretical models and experiments, and it can be duplicated in studying different models. In the end, we show the differential decay rate spectra, angular correlation spectra, and the half-life expressions.

Contents

1 Motivation and Introduction

The *Standard Model* (SM) [1] is a milestone of the development of theoretical particle physics, which describes particles and the behaviors of particles under *strong*, *weak* and *electromagnetic* forces [2]. The latter two have been unified into *electroweak* mechanism which performs a more elegant explanation for leptons [3, 4, 5]. Developed in the early 1970s, the SM has successfully become established as a well-tested physics theory after extensive precise experiments, especially high energy colliding experiments with large colliders such as LHC and LEP [6].

However, starting in the 1970s, physicists began to search for more precise symmetries to explain the physics beyond the SM (BSM) [7]. One of the phenomena, though the neutrino oscillation had not been conclusively confirmed at that time, promotes theories for massive neutrinos [8]. This phenomenon is not included in the SM theory, where neutrinos are Dirac particles and are assumed to be massless. The massive neutrinos, which are electrically neutral fermions, provide a hint that neutrinos can be Majorana particles [9]. Accordingly, the question of whether neutrinos are Majorana or Dirac particles becomes a crucial aspect in the search for new BSM physics [10]. In theoretical perspective, physicists have created several models to match the existing or upcoming experimental results, including those that are gauge invariant with Majorana particles. The most propitious candidate experiment for studying massive Majorana neutrinos is the neutrinoless double beta decay ($0\nu\beta\beta$) which indicates the violation of lepton number conservation. It is necessary to derive the decay rate of the $0\nu\beta\beta$ decays in order to put constraints on new physics parameters and to distinguish the underlying mechanism of the lepton number violation physics [11, 12]. The expressions of the decay rate depend on the choice of different models and mechanisms [13]. These predictions play a significant role in understanding the deep mechanism of neutrinoless double beta decay. However, people will only know whether the models are worthy or not once the process is observed [14]. In this thesis, we will calculate the neutrinoless double beta decay rate under the *the minimal left-right symmetric model* and give constrains on the effective Lagrangian couplings.

In chapter 2, we will talk about the mass mechanism of neutrinos and review the Dirac and Majorana type neutrino terms in the SM. A few pages are added to the last part of chapter 2 to explain the neutrino oscillation. In chapter 3, we will review the model building of the minimal left-right symmetric model (mLRSM) and derive the lepton and boson mass scales. Calculations are mainly performed in chapter 4, where we show the explicit calculation for tree-level differential decay rate under the mLRSM. We will give the numerical values of the phase space factors for ^{136}Xe, and draw the single electron kinetic energy spectrum and the angular correlation. One could also find some useful tools for doing the calculation in the Appendix I.

Prerequisite

1. We are using the natural unit [15] where

$$\hbar = c = 1 \tag{1.1}$$

where $\hbar$ is the reduced Planck constant. $\hbar = \frac{h}{2\pi}$, h is the Planck constant [16]. c is the speed of light. In this case, the electron mass is

$$m_e = 9.109 \times 10^{-31}\text{kg} \approx 0.511\text{MeV} \tag{1.2}$$

2. The relativistic 4-dimensional metric tensor we are using is:

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1.3}$$

Sometimes it is written as $\eta_{\mu\nu}, \eta^{\mu\nu}$ in Minkowski coordinates. Define gamma matrices in 4-dimensional space $\gamma^\mu = (\gamma^0, \gamma^1, \gamma^2, \gamma^3)$. They satisfies the anticommutation relation,

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = g^{\mu\nu}\mathbb{I}_{4\times 4} \tag{1.4}$$

The transformation of the upper or lower indices is through the metric acting on the gamma matrix: $\gamma_\mu = g_{\mu\nu}\gamma^\nu = \gamma_\mu = (\gamma_0, -\gamma_1, -\gamma_2, -\gamma_3)$, where the same index is so-called the "dummy index" and stands for Einstein summation [17], e.g. $\gamma_\mu\gamma^\mu = g_{\mu\nu}\gamma^\mu\gamma^\nu = g^{\mu\nu}\gamma_\mu\gamma_\nu = \gamma_0\gamma^0 + \gamma_1\gamma^1 + \gamma_2\gamma^2 + \gamma_3\gamma^3$. The upper and lower Lorentz indices are sometimes called the *contravariant* and *covariant* indices, respectively [18]. In different representations, gamma matrices have different content. Define the additional fifth gamma matrix $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ or $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$, and i is the imaginary unit. γ^5 matrix is Hermitian $\gamma^{5\dagger} = \gamma^5$ and the square equals identity $(\gamma^5)^2 = \mathbb{I}_{4\times 4}$. These can be found in quantum field theory books, e.g. [19, 20, 21].

2 Neutrino Mass

The "Neutrino" was first postulated by Wolfgang Pauli in 1930 in his letter "Liebe Radioaktive Damen und Herren" [22]. He introduced this electric neutral particle with mass similar to the electron mass to explain the energy and the momentum conservation in beta decay. In 1956, the electron neutrino was first discovered by Reines and Cowan in the inverse beta process from a nuclear reactor [23]. When the muon neutrino was later observed in pion decay in 1962 at Brookhaven National Laboratory [24, 25], the concepts of lepton families became more acceptable in theoretical models [26]. Finally in the 1970s, the combination of electroweak interaction theory and the strong interaction theory gave the birth of the SM [27, 28]. In this chapter, we will review the theories of the neutrino mass from an overview perspective instead of a historical one. We will introduce the SM and consider how we can add neutrino masses to fix the model. In order to better understand the neutrino mass, we will review the neutrino oscillation and the neutrino mixing matrix (PMNS matrix) in section 2.5. For historical review, see [29].

2.1 The Standard Model Lagrangian Revisited

In this discussion, we only take into account the electroweak interactions and leave the strong interactions for a minute, which means that the symmetry group considered here is $SU(2) \times U(1)$. More precisely, due to the unobserved right-handed current, people built up the chiral asymmetry that places the left-handed particles into doublets and right-handed particles into singlets, i.e. the symmetry group is $SU(2)_L \times U(1)_Y$. The conserved charge is $Q = I_3 + Y/2$, where I_3 is the isospin and Y is the electroweak hypercharge. The Lagrangian in this case contains the kinetic terms (gauge boson fields, fermion fields, Higgs bosons), the Yukawa terms, and the potential terms. We can write down the Lagrangian with three generations of fermions as follows (only in $SU(2)_L \times U(1)_Y$) [30],

$$\mathscr{L} = \mathscr{L}_g + \mathscr{L}_f + \mathscr{L}_H + \mathscr{L}_Y + V(\Phi) \tag{2.1}$$

where "g", "f", "H", and "Y" represent the gauge, fermions, Higgs, and Yukawa, respectively. There is no term for gluon color field interaction since we only consider the symmetry group $SU(2)_L \times U(1)_Y$. We can define each terms separately. The first terms is the gauge fixing of the bosons fields,

$$\mathscr{L}_g = -\frac{1}{4} W^{\mu\nu} W_{\mu\nu} - \frac{1}{4} B^{\mu\nu} B_{\mu\nu} \tag{2.2}$$

where W, B are gauge bosons and $W^{\mu\nu,a} = \partial^\mu W^{\nu,a} - \partial^\nu W^{\mu,a}$, $B^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu$ are the field strength tensor. The second term is the fermion kinetic term with gauge fixing, and it is defined as

$$\begin{aligned}\mathscr{L}_f = &\sum_{i=e,\mu,\tau} \overline{L_{i,L}} i \not{D} L_{i,L} + \sum_{i=1,2,3} \overline{Q_{i,L}} i \not{D} Q_{i,L} \\ &+ \sum_{i=e,\mu,\tau} \overline{l_{i,R}} i \not{D} l_{i,R} + \sum_{\substack{i=u,c,t \\ or\ i=d,s,b}} \overline{q_{i,R}} i \not{D} q_{i,R}\end{aligned} \tag{2.3}$$

and we use the standard notation $\not{D} \equiv \gamma^\mu D_\mu$, where γ^μ is the gamma matrix. The covariant derivative is given as

$$D = \partial_\mu + i g_{SM} \frac{\vec{\sigma}}{2} \cdot W_\mu + g'_{SM} \frac{Y}{2} \tag{2.4}$$

where Y is the hypercharge (one need not to be confused with the Yukawa couplings). $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are Pauli matrices. The left-handed fermion fields are,

$$L_{e,L} = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \qquad L_{\mu,L} = \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L \qquad L_{\tau,L} = \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L \tag{2.5}$$

$$Q_{1,L} = \begin{pmatrix} u \\ d \end{pmatrix}_L \qquad Q_{2,L} = \begin{pmatrix} c \\ s \end{pmatrix}_L \qquad Q_{3,L} = \begin{pmatrix} t \\ b \end{pmatrix}_L \tag{2.6}$$

while the corresponding right-handed singlets are $q^{u,d}_{j,R}$ and $l_{j,R}$ where q, l represent quarks and charged leptons respectively. u, d are up type and down type quarks. $\overline{L}$ is defined in the usual way as $\overline{L} = L^\dagger \gamma^0$. The fields with chiral indices are expressed by acting with the chiral projection operator $P_{L.R}$ on the original fields. For example, the left-hand field of an arbitrary fermion field Ψ is $P_L \Psi = \Psi_L$. The projection operator is defined through the gamma-5 matrix: $P_{L,R} = \frac{1 \mp \gamma^5}{2}$. The Higgs kinetic term is defined as,

$$\mathscr{L}_H = (D_\mu \Phi)^\dagger (D_\mu \Phi) \tag{2.7}$$

The Higgs field Φ is a doublet with hypercharge $Y_\Phi = 1$ in order to contract with the left-handed doublets in the Lagrangian. The Yukawa term is defined as

$$\begin{aligned}\mathscr{L}_Y = &- \sum_{i,j=e,\mu,\tau} \left(Y^l_{ij} \overline{L_{i,L}} \Phi l_{j,R} + Y^{l\,*}_{ij} \overline{l_{j,R}} \Phi^\dagger L_{i,L} \right) - \sum_{\substack{i=1,2,3 \\ j=u,c,t}} \left(Y^u_{ij} \overline{Q_{i,L}} \Phi q^u_{j,R} + Y^{u*}_{ij} \overline{q^u_{j,R}} \Phi^\dagger Q_{i,L} \right) \\ &- \sum_{\substack{i=1,2,3 \\ j=d,s,b}} \left(Y^d_{ij} \overline{Q_{i,L}} \Phi q^d_{j,R} + Y^{d*}_{ij} \overline{q^d_{j,R}} \Phi^\dagger Q_{i,L} \right)\end{aligned} \tag{2.8}$$

where $Y_{i,j}^{l,u,d}$ are the Yukawa couplings. The potential term contains information about the environment, or more precisely, the "vacuum". This term gives the potential field with which the other fields interact. From the simplest renormalizable "ϕ^4" toy model, we know the potential term has $O(N)$ symmetry and indicates a non-trivial vacuum expectation value (VEV) [19]. This non-trivial VEV leads to a so-called spontaneous symmetry breaking (SSB) mechanism, which automatically generates the mass of the gauge bosons. We use the ϕ^4 potential in the $O(2)$ symmetry here,

$$V(\Phi) = \frac{1}{2}\mu^2\Phi^\dagger\Phi + \frac{\lambda}{4}(\Phi^\dagger\Phi)^2 \tag{2.9}$$

where $\Phi^\dagger = (\phi_1^\dagger, \phi_2^\dagger)$. We can find global minima when the potential satisfies $\frac{\partial V}{\phi_1} = \frac{\partial V}{\phi_2} = 0$. That is,

$$|\phi_1|^2 + |\phi_2|^2 = -\frac{\mu^2}{\lambda} \tag{2.10}$$

which indicates a massless field and a massive field in the polar coordinates. Let us define the vacuum Higgs field under the operator $\hat{Q} = \hat{I}_3 + \frac{\hat{Y}}{2}$ is $\Phi = \begin{pmatrix}\phi^+ & \phi^0\end{pmatrix}^T$ and only the neutral field ϕ_0 gains the VEV due to the conservation of the electric charge. In this way we can solve the VEVs from Equation (2.10): $\Phi_0 = \begin{pmatrix}0 & \sqrt{-\frac{\mu^2}{\lambda}}\end{pmatrix}^T$. In $O(N)$ SSB, we have the VEV $\Phi_0 = (0, 0, ..., v)^T$ where there are $N-1$ zeros. The $O(N)$ symmetry is hidden and leaves only $O(N-1)$.

2.2 Higgs Mechanism and Lepton Masses

A widely accepted model about how leptons gain their masses is through the Higgs mechanism. The Higgs field Φ has a non-trivial VEV due to SSB. For convenience, we can define the Higgs field in the Lagrangian (2.1) in the unitary gauge. Since the potential (2.9) and the minima condition (2.10), we can arbitrarily choose the form

$$\Phi(x) = \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix} \tag{2.11}$$

and the expectation value is

$$\langle\Phi\rangle = \begin{pmatrix} 0 \\ v \end{pmatrix} \tag{2.12}$$

where $v = \sqrt{-\frac{\mu^2}{\lambda}}$ is the chosen solution from (2.10). If we are only interested in the masses of gauge bosons, we should introduce the Higgs field VEV into (2.7). The gauge fields $W_\mu^1, W_\mu^2, W_\mu^3, B_\mu$ turn into $W_\mu^\pm, Z_\mu, A_\mu$ after the Higgs mechanism [20].

The fermions masses are generated from Yukawa terms (2.8) with the Higgs VEV. It is proper to use the form (2.12) to contract with the $SU(2)$ gauge field W_μ in fundamental representation and the lepton doublet in one generation. When we put the VEV back into Equation (2.8), we can obtain:

$$\mathscr{L}_Y^l = -v \sum_{i,j=e,\mu,\tau} \overline{l_{i,L}} Y_{ij}^l l_{j,R} + h.c. \tag{2.13}$$

where we only write down the Yukawa term for leptons (the superscript $l \equiv$ lepton). The lepton states have conventionally three generations, and the Yukawa couplings then form a 3×3 matrix. To diagonalize the Yukawa matrix, we need the transformation,

$$l_{L,R}^m = U_{L,R}^{l\dagger} l_{L,R} = \begin{pmatrix} e^m \\ \mu^m \\ \tau^m \end{pmatrix}_{L,R} \tag{2.14}$$

and the transformation of the conjugate lepton state $\overline{l_{L,R}^m} = \overline{l_{L,R}} U_{L,R}^l$. We have used the superscript m to represent the mass state leptons. In this way, the Yukawa matrix transforms as $U_L^{l\dagger} Y^l U_R^l = Y^{lM}$, where Y^{lM} is a 3×3 diagonal matrix, and the entries are $y_{e,\mu,\tau}^l$. The small l in the superscript is for charged leptons. The pure lepton mass term then reduces to the form

$$\mathscr{L}_{m_l}^D = -v \sum_{\alpha=e,\mu,\tau} y_\alpha^l \bar{l}_\alpha l_\alpha \tag{2.15}$$

where $l_\alpha \equiv l_{\alpha L} + l_{\alpha R}$. We use l_α instead of $l_{e,\mu,\tau}^m$ (to get rid of m) to simplify the expression. In the SSB and Higgs mechanism processes, only the charged leptons gain masses through Yukawa couplings and non-trivial VEVs. The neutrinos are Dirac type and remain massless [30]. In general, we introduce a Higgs field $h(x)$ with zero VEV in the position basis, see Equation (2.11). Then the full Lagrangian of the lepton mass term is

$$\mathscr{L}_{M_l}^D = -v \sum_{\alpha=e,\mu,\tau} y_\alpha^l \bar{l}_\alpha l_\alpha - \sum_{\alpha=e,\mu,\tau} y_\alpha^l \bar{l}_\alpha l_\alpha h(x) \tag{2.16}$$

where the second term is the Higgs-lepton interaction vertex. It shows that the vertex should contain a coefficient proportional to the Yukawa couplings, or more detailed, the mass of the leptons.

2.3 Dirac Mass for Neutrinos

Neutrinos are neutral fermions. In a very natural way of thinking, we can add a right-handed neutrino singlet similar to the right-handed charged lepton and the neutrino gains mass through the same Higgs mechanism procedure (with the Yukawa couplings) as we did in the last section. This does not change the symmetry $SU(2)_L\times U(1)_Y$ of the theory and it is so-called the fixing of the SM for the neutrino mass, or the Dirac neutrino mass. The Lagrangian is enlarged by the right-handed neutrino singlet term,

$$\mathscr{L}^D_{\text{leptonmass}} = -\sum_{i,j=e,\mu,\tau} Y^l_{ij}\overline{L_{i,L}}\Phi l_{j,R} + Y^\nu_{ij}\overline{L_{i,L}}\tilde{\Phi}\nu_{j,R} + h.c. \tag{2.17}$$

where we introduce a new Higgs field $\tilde{\Phi} \equiv i\sigma_2\Phi^*$ with hypercharge $Y_{\tilde{\Phi}} = -1$. $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ belongs to the Pauli matrix. $\tilde{\Phi}$ provides the same potential as Φ, but with VEV $\langle\tilde{\Phi}\rangle = \begin{pmatrix} v & 0 \end{pmatrix}^T$ instead. The neutrino singlet ν_R is defined as

$$\nu_R = \begin{pmatrix} \nu_{eR} \\ \nu_{\mu R} \\ \nu_{\tau R} \end{pmatrix} \tag{2.18}$$

Then the mass term becomes,

$$\mathscr{L}^D_{m_l} = -(v+h(x))\sum_{\alpha=e,\mu,\tau} y^l_\alpha \bar{l}_\alpha l_\alpha - (v+h(x))\sum_{i=1,2,3} y^\nu_i \overline{\nu}_i\nu_i \tag{2.19}$$

where the different Yukawa couplings y^ν_i generate the neutrino masses. y^ν_i are entries of the diagonalized Yukawa matrix $Y^{\nu M}$. The neutrino field ν_i is $\nu_i \equiv \nu_{iL} + \nu_{iR}, i = 1,2,3$. Equation (2.19) differs from Equation (2.16) only by adding the corresponding neutrino term. The subscript $1,2,3$ stands for the neutrino mass eigenstates, and we have the flavor-mass eigenstates defined via:

$$\nu_{lL} = \sum_{i=1}^{3} U^L_{li}\nu_{iL} \qquad\qquad \nu_{lR} = \sum_{i=1}^{3} U^R_{li}\nu_{iR} \tag{2.20}$$

where the transformation matrix $U^{L,R}$ are unitary and so-called the PMNS mixing matrix (the Pontecorvo-Maki-Nakagawa-Sakata matrix) [31, 32]. The right-handed neutrino is named the 'sterile' neutrino, where B.Pontecorvo first gave this name for its almost unobservable property [32]. Sterile neutrinos have zero isospin since they are $SU(2)_L$ singlets. Their hypercharge also need to be zero since the hypercharge of the Higgs field and the charged lepton field have already matched. The sterile neutrino is neutral with the definition of Q [32]. This result leads to a fact that sterile neutrinos do not participate in electroweak interactions. Also, they are obviously leptons and hence do not take part in the strong interactions. Thus, in SM, $SU(3)_C\times SU(2)_L\times U(1)_Y$, the sterile neutrinos only interact with the Higgs field and the active neutrinos through Yukawa interactions. One important question we should ask here is if the Dirac mass term is gauge invariant or not. First, Let us consider a local

Field	$SU(2)_L$	$U(1)_Y$	Q
$L_L = \begin{pmatrix} \nu_L \\ l_L \end{pmatrix}$	2	-1	$\begin{pmatrix} 0 \\ -1 \end{pmatrix}$
l_R	1	-2	-1
ν_R	1	0	0
Φ	2	1	1
$\tilde{\Phi}$	2	-1	-1

Table 2.1: Electroweak fields assignment in the SM

gauge transformation $G(\theta_a(x), \eta(x))$ under $SU(2)_L \times U(1)_Y$. G is a 3+1 parameters transformation where the $SU(2)_L$ local gauge is $\theta_a(x) = (\theta_1(x), \theta_2(x), \theta_3(x))$. The gauge transformation G is

$$G(\theta_a(x), \eta(x)) = \mathrm{e}^{i\theta_a(x)\frac{\tau_a}{2} + i\eta(x)\frac{Y}{2}} \tag{2.21}$$

where $\frac{\tau_a}{2}$ are the generators of the $SU(2)$ group. τ_a is actually the Pauli matrix. With the lepton hypercharge $Y_{lL} = -1, Y_{lR} = -2, Y_{\nu R} = 0$, the gauge transformations are given as

$$\begin{aligned} L^g_{lL} &= \mathrm{e}^{i\frac{\theta_a(x)}{2}\tau_a - i\frac{1}{2}\eta(x)} L_{lL} & l^g_R &= \mathrm{e}^{-i\eta(x)} l_R & \nu^g_{lR} &= \nu_{lR} \\ \overline{L}^g_{lL} &= \overline{L}_{lL}\mathrm{e}^{-(i\frac{\theta_a(x)}{2}\tau_a - i\frac{1}{2}\eta(x))} & \overline{l}^g_R &= \overline{l}_R \mathrm{e}^{i\eta(x)} & \overline{\nu}^g_{lR} &= \overline{\nu}_{lR} \end{aligned} \tag{2.22}$$

The gauge transformations of the Higgs field are

$$\Phi^g = \mathrm{e}^{i\frac{\theta_a(x)}{2}\tau_a + i\frac{1}{2}\eta(x)}\Phi \qquad \tilde{\Phi}^g = \mathrm{e}^{i\frac{\theta_a(x)}{2}\tau_a - i\frac{1}{2}\eta(x)}\tilde{\Phi} \tag{2.23}$$

Combining with (2.22) and (2.23), the gauge transformation of the Lagrangian (2.17) is

$$\begin{aligned} \mathscr{L}^{Dg}_{\text{leptonmass}} =& -\sum_{i,j=e,\mu,\tau} Y^l_{ij}\overline{L}^g_{i,L}\Phi^g l^g_{j,R} + Y^\nu_{ij}\overline{L}^g_{i,L}\tilde{\Phi}^g\nu^g_{j,R} + h.c. \\ =& -\sum_{i,j=e,\mu,\tau} Y^l_{ij}\overline{L}_{i,L}\mathrm{e}^{-(i\frac{\theta_a(x)}{2}\tau_a - i\frac{1}{2}\eta(x))}\mathrm{e}^{i\frac{\theta_a(x)}{2}\tau_a + i\frac{1}{2}\eta(x)}\Phi\mathrm{e}^{-i\eta(x)} l_{j,R} \\ & + Y^\nu_{ij}\overline{L}_{i,L}\mathrm{e}^{-(i\frac{\theta_a(x)}{2}\tau_a - i\frac{1}{2}\eta(x))}\mathrm{e}^{i\frac{\theta_a(x)}{2}\tau_a - i\frac{1}{2}\eta(x)}\tilde{\Phi}\nu_{j,R} + h.c. \\ =& \mathscr{L}^D_{\text{leptonmass}} \end{aligned} \tag{2.24}$$

The lepton mass Lagrangian we build in this way is $SU(2)_L \times U(1)_Y$ local gauge invariant. For explicit reading, these field assignments are shown in Table 2.1.

2.4 Majorana Mass and the Seesaw Mechanism

Apart from the Dirac type particle, neutrinos could also be Majorana particles. The Majorana particle is defined in such a way that the particle and the antiparticle coincide [9, 33]. Let us first discuss the antiparticle expression. Antiparticles have opposite charge number and chirality, while remain the same mass with respect to the correspondent particle. In order to transit transfer between particles and antiparticles, we can define the particle-antiparticle operator as $\hat{C}$ [34].

$$\hat{C} : \Psi \to \Psi^c \equiv C\overline{\Psi}^T \tag{2.25}$$

where the charge conjugate matrix C satisfies:

$$C(\gamma^\mu)^T C^{-1} = -\gamma^\mu \qquad C(\gamma^5)^T C^{-1} = \gamma^5 \qquad C^\dagger C = \mathbb{1} \tag{2.26}$$

for Dirac spinors. These relations ensure that the field Ψ^c is Lorentz covariant [35]. One important point as Evgeny Kh. Akhmedov mentioned in his paper [34] is that, one should not be confused with the charge conjugate operator C and the particle-antiparticle operator $\hat{C}$. Although they are equivalent for Dirac and Majorana spinors, they are not the same for chiral fields. The particle-antiparticle conjugate operator will change the chirality, but the charge conjugate operator does not. Find the details here [34]. The charge conjugate operator acting on the fields of Dirac spinors in this way is [19]

$$\begin{aligned} C\Psi(x)C &= -i(\overline{\Psi}\gamma^0\gamma^2)^T \\ C\overline{\Psi}(x)C &= (-i\gamma^0\gamma^2\Psi)^T \end{aligned} \tag{2.27}$$

which means that the charge conjugate operator satisfies the form $C = -i\gamma^0\gamma^2$ in the Dirac representation. The definition and the properties of C tell us that only neutral particles can be Majorana type. For the Dirac type particles masses, only the $\overline{\Psi_L}\Psi_R + \overline{\Psi_R}\Psi_L$ term survives because of the projection operator property $P_L P_R = 0$. One question is what the mass term looks like for Majorana fields. For example, we have the particle-antiparticle conjugate for left-handed field,

$$\begin{aligned} \Psi_L^c &= (P_L\Psi)^c = C\overline{(P_L\Psi)}^T = C[(P_L\Psi)^\dagger\gamma^0]^T = C[\Psi^\dagger\gamma^0 P_R]^T \\ &= CP_R\overline{\Psi}^T = P_R C\overline{\Psi}^T = (\Psi^c)_R = \Psi_R \end{aligned} \tag{2.28}$$

where the $\hat{C}$ operator transforms the left-handed chirality to the right-handed chirality. So, for Majorana neutrinos, we could have a mass term formed by $\overline{\nu_L^c}\nu_L + h.c.$. We can build the Lagrangian by only using ν_L. Combined with the kinetic term, we have the Lagrangian for Majorana neutrinos,

$$\mathscr{L}_\nu^M = \sum_i \left[\frac{1}{2}(\overline{\nu_{iL}} i\partial\!\!\!/ \nu_{iL} + \overline{\nu_{iL}^c} i\partial\!\!\!/ \nu_{iL}^c) - \frac{1}{2}m_i^M(\overline{\nu_{iL}^c}\nu_{iL} + \overline{\nu_{iL}}\nu_{iL}^c)\right] \tag{2.29}$$

where we use the fact $(\partial^\mu \overline{\nu_{iL}^c}) i\gamma_\mu \nu_{iL}^c = -\overline{\nu_{iL}^c} i\gamma_\mu \partial^\mu \nu_{iL}^c$ from the partial integration with the boundary condition that $\int d^4x \partial^\mu (\overline{\nu_{iL}^c}) i\gamma_\mu \nu_{iL}^c) = 0$ [36]. We only think of the singlet Majorana neutrinos which do not interact with the gauge field doublet. i is summed over generations. m_i are the entries of the diagonalized $M_{3\times 3}^M$ matrix. The fields follow the same transformation as the Dirac neutrinos, see Equation (2.20). Define a neutrino field $\nu_i^M = \nu_{iL} + \nu_{iL}^c$, and this is obviously a Majorana field $\nu_i^M = (\nu_i^M)^c$. Then the pure Majorana mass term goes to

$$\mathscr{L}_{m_\nu}^M = -\frac{1}{2}\sum_{i=1}^{3} m_i^M \overline{\nu_i^M} \nu_i^M \tag{2.30}$$

where we consider the conventional three generations. Detailed discussion of the spinors under particle-antiparticle operator is given in section (4.3.1) and in Appendix E when we derive the neutrino propagators. This Majorana fields in (2.29) will not be invariant under a global $U(1)$ transformation $\mathrm{e}^{i\Lambda}$.

$$\nu_L^M \xrightarrow{\Lambda} \mathrm{e}^{i\Lambda}\nu_L^M \qquad\qquad \nu_L^c \xrightarrow{\Lambda} (\mathrm{e}^{i\Lambda}\nu_L^M)^c = \mathrm{e}^{-i\Lambda} v_L^c \tag{2.31}$$

The conjugate then transforms as $\overline{\nu_L^c} \xrightarrow{\Lambda} \overline{\nu_L^c}\mathrm{e}^{i\Lambda}$. We can write the Majorana mass term in Equation (2.29) under this arbitrary global transformation.

$$-\frac{1}{2} m_i^M (\overline{\nu_{iL}^c}\nu_{iL} + \overline{\nu_{iL}}\nu_{iL}^c) \xrightarrow{\Lambda} \frac{1}{2} m_i^M (\overline{\nu_{iL}^c}\mathrm{e}^{i\Lambda}\mathrm{e}^{i\Lambda}\nu_{iL} + \overline{\nu_{iL}}\mathrm{e}^{-i\Lambda}\mathrm{e}^{-i\Lambda}\nu_{iL}^c) \tag{2.32}$$

This means the Majorana neutrino term is not globally invariant under the $U(1)$ symmetry. The Majorana term (2.29) violates the conservative charge number which is the lepton number given by the Nöther theorem, i.e. the lepton number is not conserved with the Majorana neutrino term. Let us consider about what if we put the Dirac term and the Majorana term together. One can find it in many text books [30, 35, 36]. Combining Equation (2.19) and (2.30), we have:

$$\mathscr{L}_{m_\nu}^{D+M} = -\sum_{i}^{3} (m_i^D \overline{\nu_i^D} \nu_i^D + \frac{1}{2} m_i^M \overline{\nu_i^M} \nu_i^M) \tag{2.33}$$

In Equation (2.30), the Majorana field is expressed only in the left-handed fields ν_L, ν_L^c. In order to complete the combination with the Dirac neutrino consistently, we must add the right-handed correspondence, i.e. ν_R. Then we can form the Dirac neutrino and the Majorana neutrino with the same chirality to a doublet,

$$\mathscr{L}_{m_\nu}^{D+M} = -\overline{\nu_L^D} M^D \nu_R^D - \frac{1}{2}(\overline{\nu_L^M} M_L^M (\nu_L^M)^c + \overline{(\nu_R^M)^c} M_R^M \nu_R^M) + h.c. \tag{2.34}$$

where M^D and M^M are non-diagonalized mass matrices for the Dirac and Majorana neutrino. Define the doublet and its conjugate,

$$n_L = \begin{pmatrix} \nu_L \\ \nu_R^c \end{pmatrix} \qquad \overline{n_L} = \begin{pmatrix} \overline{\nu_L} & \overline{\nu_R^c} \end{pmatrix} \qquad n_L^c = \begin{pmatrix} \nu_L^c \\ \nu_R \end{pmatrix} \tag{2.35}$$

with this we form the two types of neutrino field into a matrix term.

$$\begin{aligned} \mathscr{L}_{m_\nu}^{D+M} &= -\frac{1}{2}\overline{n_L} M^{D+M} n_L^c + h.c. \\ &= -\frac{1}{2}\begin{pmatrix} \overline{\nu_L} & \overline{\nu_R^c} \end{pmatrix} \begin{pmatrix} M_L^M & M^D \\ (M^D)^T & M_R^M \end{pmatrix} \begin{pmatrix} \nu_L^c \\ \nu_R \end{pmatrix} + h.c. \end{aligned} \tag{2.36}$$

where M^D is supposed to be real and the term $\overline{\nu_R^c}(M^D)^T\nu_L^c$ is equivalent to $\overline{\nu_L}M^D\nu_R$.

$$\overline{\nu_R^c}(M^D)^T\nu_L^c = -(\overline{\nu_L}C^T M^D (C^{-1})^T \nu_R)^T = -(\overline{\nu_L}CM^D C^{-1}\nu_R)^T = \overline{\nu_L}M^D\nu_R \tag{2.37}$$

where we use $C^T = -C, (C^{-1})^T = -C^{-1}$ and the expression:

$$\overline{\nu_L^c} = \overline{C\overline{\nu^T}} = (\overline{\nu_L}^T)^\dagger C^\dagger \gamma^0 = [(\nu_L^T)^*\gamma^0]^\dagger C^\dagger\gamma^0 = \nu_L^T\gamma^0 C^{-1}\gamma^0 = -\nu_L^T C^{-1} \tag{2.38}$$

The block matrix in Equation (2.36) is a 6×6 matrix. There are not only mixing of generations inside each 3×3 blocks, but also mixing between the Dirac neutrinos and the Majorana neutrinos. After the diagonalization, we should end in:

$$\begin{aligned} \mathscr{L}_{m_\nu}^{D+M} &= \frac{1}{2}\overline{U^\dagger n_L} m^{D+M}(U^\dagger n_L)^c + h.c. = -\frac{1}{2}\overline{\nu^{D+M}} m^{D+M}\nu^{D+M} \\ &= -\frac{1}{2}\sum_{i=1}^{6} m_i^{D+M}\overline{\nu_i}\nu_i \end{aligned} \tag{2.39}$$

where $\nu^{D+M} = n_L^{(m)} + n_L^{(m)^c} = (\nu_1, \nu_2, \ldots, \nu_6)^T$ is defined in the mass eigenstates. The transformation matrix U is also a 6×6 matrix. The mixed n_L field transforms as $n_{lL} = \sum_{i=1}^{6} U_{li} n_{iL}^{(m)}$. The Dirac-Majorana mixed Lagrangian is definitely not global $U(1)$ invariant, since we have shown that the Majorana term already violates the global symmetry, see (2.32). In the SM, it is not possible to find a global invariant Majorana term. But physicists have made such terms in the BSM, for example in the left-right symmetric model [37]. Equation (2.39) has shown the Dirac and Majorana es in the mass eigenstate. However, the mass expressions m_i^{D+M} could be derived in a more explicit structure. A physically instructive way is to consider the simplest one generation case. Let us start from Equation (2.34) and omit the superscript "M" of the Majorana case for simplicity. The Lagrangian in this case becomes

$$\mathscr{L}_m^{D+M} = -m_D\overline{\nu_L}\nu_R - \frac{1}{2}m_L\overline{\nu_L}\nu_L^c - \frac{1}{2}m_R\overline{\nu_R^c}\nu_R + h.c. \tag{2.40}$$

where m_L, m_R, m_D are assumed to be real. Then we have the mass matrix:

$$M^{D+M} = \begin{pmatrix} m_L & m_D \\ m_D & m_R \end{pmatrix} \tag{2.41}$$

Use the eigenfunction $M^{D+M}A = \lambda A$ to solve the eigenvalues in order to diagonalize the mass matrix. It is equivalent to find the solutions of the quadratic function $\lambda^2 - (m_L + m_R)\lambda - m_D^2 + m_L m_R = 0$ which comes from the determinant $det(M^{D+M} - \lambda \mathbb{1}_{2\times 2}) = 0$. The eigenvalues are denoted as m_1, m_2, and the expressions are

$$\begin{aligned} m_1 &= \frac{1}{2}(m_L + m_R) - \frac{1}{2}\sqrt{(m_R - m_L)^2 + 4m_D^2} \\ m_2 &= \frac{1}{2}(m_L + m_R) + \frac{1}{2}\sqrt{(m_R - m_L)^2 + 4m_D^2} \end{aligned} \tag{2.42}$$

Another way to diagonalizing (2.41) is to transform the neutrino flavor states into mass eigenstates. Any real symmetric 2×2 matrix can be diagonalized through the similarity transformation of an unitary matrix U, i.e. $m = U^\dagger M^{D+M} U$. Then we can easily choose the transformation matrix proportional to a rotational matrix in 2-dimension. This constrains the matrix to an orthogonal matrix, and then gives $m = U^T M^{D+M} U$, where $U = \eta^{\frac{1}{2}} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ and $\eta^2 = 1$ is the proportional factor. Explicitly, we have

$$\begin{aligned} &U^T M^{D+M} U = \eta^2 \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} m_L & m_D \\ m_D & m_R \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}(m_L + m_R) - \frac{1}{2}(m_R - m_L)\cos 2\theta - m_D \sin\theta & -\frac{1}{2}(m_R - m_L)\sin 2\theta + m_D \cos 2\theta \\ -\frac{1}{2}(m_R - m_L)\sin 2\theta + m_D \cos 2\theta & \frac{1}{2}(m_L + m_R) + \frac{1}{2}(m_R - m_L)\cos 2\theta + m_D \sin\theta \end{pmatrix} \end{aligned} \tag{2.43}$$

where the mixing angle θ should satisfy the condition that makes the off-diagonal entries equal zero, that is, $-\frac{1}{2}(m_R - m_L)\sin 2\theta + m_D \cos 2\theta = 0$. Then we have

$$\tan 2\theta = \frac{2m_D}{m_R - m_L} \tag{2.44}$$

Compared to Equation (2.42), we should have

$$\frac{1}{2}\sqrt{(m_R - m_L)^2 + 4m_D^2} = \frac{1}{2}(m_R - m_L)\cos 2\theta + m_D \sin\theta \tag{2.45}$$

which gives a simple choice of condition for $\cos 2\theta$ and $\sin 2\theta$.

$$\sin 2\theta = \frac{2m_D}{\sqrt{(m_R - m_L)^2 + 4m_D^2}}; \qquad \cos 2\theta = \frac{m_R - m_L}{\sqrt{(m_R - m_L)^2 + 4m_D^2}} \tag{2.46}$$

The transformation of the neutrino states is expressed explicitly as

$$\begin{aligned}\nu^{D+M} &= U^\dagger n_L + (U^\dagger n_L)^c \\ &= \eta^{\frac{1}{2}} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^\dagger \begin{pmatrix} \nu_L \\ \nu_R^c \end{pmatrix} + \eta^{\frac{1}{2}} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^\dagger \begin{pmatrix} \nu_L^c \\ \nu_R \end{pmatrix}\end{aligned} \tag{2.47}$$

and, e.g. for the left-handed part, we have

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_L \\ \nu_R^c \end{pmatrix} = \begin{pmatrix} \sqrt{\eta_1}\nu_{1L} \\ \sqrt{\eta_2}\nu_{2L} \end{pmatrix} \tag{2.48}$$

and this gives

$$\begin{aligned}\nu_L &= \cos\theta\sqrt{\eta_1}\nu_{1L} + \sin\theta\sqrt{\eta_2}\nu_{2L} \\ \nu_R^c &= -\sin\theta\sqrt{\eta_1}\nu_{1L} + \cos\theta\sqrt{\eta_2}\nu_{2L}\end{aligned} \tag{2.49}$$

where ν_{1L} and ν_{2L} are mass eigenstates. It is necessary to say that the relation (2.44) still works for pure Majorana cases with two flavors mixing. For instance, in the ν_μ, ν_τ case, we would have the mass matrix $M^M = \begin{pmatrix} m_{\mu\mu} & m_{\mu\tau} \\ m_{\tau\mu} & m_{\tau\tau} \end{pmatrix}$, which is similar with the matrix (2.41). The mixing angle $\theta_{\mu\tau}$ between ν_μ and ν_τ satisfies $\tan 2\theta = \frac{2m_{\mu\tau}}{m_{\tau\tau} - m_{\mu\mu}}$. The flavor eigenstates $\nu_{\mu L}, \nu_{\tau L}$ have the same expressions in terms of the mass eigenstates as in (2.49).

At the Tokyo conference in 1981, the Japanese physicist Tsutomu Yanadiga proposed a mass mechanism called the "seesaw mechanism" [38]. This mechanism automatically gives an explanation of the smallness of the neutrino mass with respect to the charge-lepton mass in the corresponding generation by introducing heavy right-handed Majorana neutrinos. Now introduce the main assumptions of the easiest seesaw mechanism [36]: 1) The Dirac type neutrino mass is much smaller than the right-handed Majorana neutrino mass $m_D \ll m_M$; 2) the left-handed Majorana neutrino should result in zero mass $m_L^M = 0$. The mathematical reason for these assumptions is that from the mass Equation (2.19) and (2.30), there is no constraint for the right-handed neutrino mass m_R, and the Dirac neutrino is proportional to the Higgs VEV. Physically, ν_R is singlet with zero hypercharge, and its mass is arbitrary in the SM symmetry. Let us begin with the one generation. Using these two assumptions, we can reduce the mass expression in (2.42).

$$\begin{aligned}m_{1,2} &= \frac{1}{2}(m_R + m_L) \mp \frac{1}{2}\sqrt{(m_R - m_L)^2 + 4m_D^2} \\ &= m_R[\frac{1}{2}(1 + \frac{m_L}{m_R}) \mp \frac{1}{2}\sqrt{\frac{(m_R - m_L)^2}{m_R^2} + \frac{4m_D^2}{m_R^2}}] \\ &\approx m_R[\frac{1}{2}(1 + \frac{m_L}{m_R}) \mp \frac{1}{2}(1 + \frac{m_D^2}{m_R^2} - \frac{m_L}{m_R})]\end{aligned} \tag{2.50}$$

take the absolute value,

$$\begin{aligned} m_1 &\approx \frac{m_D^2}{m_R} \ll m_D \\ m_2 &\approx m_R \gg m_D \end{aligned} \tag{2.51}$$

The mixing angle also reduces under the assumptions,

$$\tan 2\theta = \frac{2m_D}{m_R - m_L} \approx \frac{2m_D}{m_R} \ll 1 \tag{2.52}$$

use the approximation if $\theta \ll 1, \sin\theta \approx \tan\theta \approx \theta$,

$$\Rightarrow \theta \approx \frac{m_D}{m_R} \ll 1 \tag{2.53}$$

insert the angle into Equation (2.49), the neutrino states have the form,

$$\begin{aligned} \nu_L &\approx \sqrt{\eta_1}\nu_{1L} + \frac{m_D}{m_R}\sqrt{\eta_2}\nu_{2L} \\ \nu_R^c &\approx -\frac{m_D}{m_R}\sqrt{\eta_1}\nu_{1L} + \sqrt{\eta_2}\nu_{2L} \end{aligned} \tag{2.54}$$

If we set $\eta_1 = -1, \eta_2 = 1$, it shows how the Dirac and Majorana states relate to the heavy neutrino states (real part). The diagonalization of the mass matrix has a brief picture:

$$\begin{pmatrix} m_L & m_D \\ m_D & m_R \end{pmatrix} \xrightarrow{dia} \begin{pmatrix} \frac{m_D^2}{m_R} & 0 \\ 0 & m_R \end{pmatrix} \tag{2.55}$$

the two eigenvalues satisfy $m_1 m_2 = m_D^2$ with one mass goes higher and the other one goes lower. It looks like a "seesaw". The mathematics behind this is that any 2×2 block matrix whose off-diagonal parts are much smaller than one of the diagonal parts will end in two "seesaw" like eigenvalues that one of them is larger than the other one. Now let us go to three generations and suppose all three generations of the left-handed Majorana neutrinos have zero mass. The mass matrix is

$$M^{D+M} = \begin{pmatrix} 0 & M^D \\ (M^D)^T & M_R \end{pmatrix} \tag{2.56}$$

where M_D and M_R are 3×3 matrices. For the matrix M_R, we can see that

$$\overline{\nu_R} M_R \nu_R^c = (\overline{\nu_R} M_R \nu_R^c)^T = \overline{\nu_R} C^T M_R^T \overline{\nu_R}^T = \overline{\nu_R} M_R^T \nu_R^c \tag{2.57}$$

which leads to

$$M_R = M_R^T \tag{2.58}$$

To diagonalize this matrix (2.56), we can introduce a unitary block matrix $U = \begin{pmatrix} \mathbb{1}_{3\times3} & W_1 \\ W_2 & \mathbb{1}_{3\times3} \end{pmatrix}$ and diagonalize through $m = U^T M^{D+M} U$. W_1, W_2 are two 3 by 3 matrices that satisfy $W_1 W_1^\dagger = W_2 W_2^\dagger = \mathbb{1}, W_1 = -W_2^\dagger$ due to $UU^\dagger = \mathbb{1}$.

$$\begin{aligned} &\begin{pmatrix} 1 & W_1 \\ W_2 & 1 \end{pmatrix} \begin{pmatrix} 0 & M^D \\ (M^D)^T & M_R \end{pmatrix} \begin{pmatrix} 1 & W_2^\dagger \\ W_1^\dagger & 1 \end{pmatrix} \\ &= \begin{pmatrix} W_1(M^D)^T + (M^D + W_1 M_R) W_1^\dagger & W_1 (M^D)^T W_2^\dagger + M^D + W_1 M_R \\ (M^D)^T + W_2 M^D + M_R W_1^\dagger & (M^D)^T + W_2 M^D + M_R \end{pmatrix} \end{aligned} \tag{2.59}$$

and the off-diagonal block should be zero. That is

$$W_2 = -M_R^{-1}(M^D)^T \pm ... \approx -M_R^{-1}(M^D)^T \tag{2.60}$$

where ... is the rest part and may have the form $\sqrt{((M^D)^T M^D)^2/(M_R^2) - 4(M^D(M^D)^T)(M^D)^T)/M_R}$ which is imaginary if M_R is much smaller than M^D. Using the relation between W_1 and W_2, we obtain

$$W_1 = -W_2^\dagger = [(M^D)^T]^\dagger (M_R^{-1})^\dagger \tag{2.61}$$

So the diagonalized matrix under the same approximation is:

$$m \approx \begin{pmatrix} -M^D M_R^{-1} (M^D)^T & 0 \\ 0 & M_R \end{pmatrix} \tag{2.62}$$

After the seesaw mechanism, we have two kinds of neutrinos with light masses and heavy masses. This implies that there should exist another kind of particle with a very heavy mass to explain the smallness of neutrinos mass neutrino masses in the SM. It is easy to show, if the mass scale of the light Majorana neutrino is $\sim 10^{-6}$eV and the lepton mass scale is around electron mass $m_D \sim 0.511$MeV, the mass scale of M_R is approximately 2.6×10^8GeV. This seesaw mechanism is historically called "the type I seesaw". Type I seesaw mechanism is characterised by a Higgs-lepton pair with a heavy neutrino singlet. The singlet does not have mass constraints so that we would have $M_R \gg m_D$. Furthermore, there are type II and type III seesaw mechanisms. The type II seesaw mechanism introduces a Higgs bidoublet and two Higgs scalar triplets for Higgs-lepton pairs with a right-handed doublet that forms $SU(2)_R$ symmetry. This is mostly used in the LRSM (left-right symmetric model) and we will discuss it in detail in chapter 3. In the type III seesaw mechanism, fermion triplets have been taken into account. To revisit the popular three types, see these: [39, 40, 41]. Moreover, there are other kinds of seesaw types, e.g. inverse seesaw mechanism, linear seesaw mechanism (review here [40] and its references), universal seesaw mechanism [42, 43], etc.

2.5 Neutrino Oscillations

Neutrino oscillation provides the experimental fact that the neutrinos are massive particles. Takaaki Kajita and Arthur B. McDonald won the 2015 physics Nobel price due to their pioneering observations of neutrino oscillations [44]. The experiments of searching neutrino oscillation has been underway for several years. The measurements of the solar electron neutrino flux was started from the 1970s in the Homestake Chlorine Detector [45]. Lederman *et al*. first discovered the muon neutrino in 1962 in the Brookhaven experiment [25]. This provides solid evidence of the theory of mixing flavor states [31]. The theory of neutrino oscillation was first proposed by Pontecorvo in the 1950s, where he considered it as an analogy to kaon oscillations [46, 47]. The theory we widely use today for neutrino oscillation is the plane wave approximation standard theory that was developed by Eliezer and Swift, Fritzsch and Minkowski, Bilenky and Pontecorvo from 1975 to 1976 [30, 48, 49, 50].

In this section, we will briefly derive the standard theory of neutrino oscillation and discuss the mixing matrix. Let us begin with the leptonic charge current

$$\begin{aligned} j^{cc}_{\alpha L}(x) &= \sum_l \overline{l_{lL}}(x)\gamma^\alpha \nu_{lL}(x) \\ &= \sum_i \overline{l_{iL}}(x) U^{l\dagger}_L \gamma^\alpha U^\nu_L \nu_{iL}(x) \\ &= \sum_i \overline{l_{iL}}(x)\gamma^\alpha U \nu_{iL}(x) \end{aligned} \tag{2.63}$$

where the leptonic charge current $j^{cc}_\alpha(x)$ is defined from the gauge boson-lepton interaction term,

$$\mathscr{L}^{CC}_{lepton} = -\frac{g_{SM}}{2} j^{cc}_{\alpha L}(x) W^+_\alpha + h.c. \tag{2.64}$$

where we have used (2.14) and (2.20) in (2.63). The lepton transformation matrices are distinguished by the use of the superscript l, ν. The relation between neutrino flavor states and mass states is (2.20). Ignoring the chirality and using the Dirac notation, we have the following.

$$|\nu_l\rangle = \sum_i^3 U_{li} |\nu_i\rangle \qquad\qquad (l = e, \mu, \tau) \tag{2.65}$$

where we consider the conventional three generations. The eigenstates are orthonormal $\langle \nu_{l/i} | \nu_{l'/j} \rangle = \delta_{ll'/ij}$. Since the mass operator and Hamiltonian operator are commute, they have the same eigenstates. We have the eigenfunction

$$\mathscr{H} |\nu_i\rangle = E_i |\nu_i\rangle \tag{2.66}$$

Inserting this into the Schrödinger equation and assuming the plane wave time evolution, we can obtain the states at some time (t)

$$|\nu_l(t)\rangle = \sum_i^3 U_{li} \mathrm{e}^{-iE_i t} |\nu_i\rangle \tag{2.67}$$

Then we can also use the transformation (2.20) to express the mass states on the basis of flavor eigenstates. Taking into account the unitary matrix U_{li}, we have

$$|\nu_l(t)\rangle = \sum_{l'=e,\mu,\tau} \big(\sum_i^3 U_{li} \mathrm{e}^{-iE_i t} U^*_{l'i}\big) |\nu_{l'}\rangle \tag{2.68}$$

Equation (2.68) gives the relation between two flavor representations with time evolution from $0 \to t$. Therefore, with the time flow $t > 0$, the massive neutrino flavor states become the superposition of all the flavor states. Additionally, there is neutrino mixing if the transformation matrix U is non-diagonal. The transition probability from the state l to l' in time period t is then

$$P_{\nu_l \to \nu_{l'}}(t) = |\langle \nu_{l'} | \nu_l(t) \rangle|^2 = \sum_{i,j}^3 U_{li} U^*_{l'i} U^*_{lj} U_{l'j} \mathrm{e}^{-i(E_i - E_j)t} \tag{2.69}$$

If we consider the small mass of the active neutrinos, the dispersion relation of the neutrino will be

$$E = |\vec{p}|(1 + \frac{m_i^2}{2|\vec{p}|} + o(2)) \approx E + \frac{m_i^2}{2E} \tag{2.70}$$

the energy difference is

$$\begin{aligned} E_i - E_j &\approx \frac{\Delta m_{ij}^2}{2E} \\ \Delta m_{ij}^2 &\equiv m_i^2 - m_j^2 \end{aligned} \tag{2.71}$$

Usually, the propagation time is not obtained in the oscillation experiments; instead, the distance between the source and the detector is measured. In the natural unit, the propagating time of an ultra-relativistic neutrino is approximately the propagating distance $t \approx \frac{L}{c} = L$. Finally, the transition probability goes to

$$P_{\nu_l \to \nu_{l'}}(L, E) = \sum_{i,j}^3 U_{li} U^*_{l'i} U^*_{lj} U_{l'j} \mathrm{e}^{-i\frac{\Delta m_{ij}^2 L}{2E}} \tag{2.72}$$

This shows that the phase shift of different flavor states determines the oscillation. The probability (2.72) also yields that the masses observed from the oscillation experiments are only the mass scales of squared-mass difference. The different orders of mass scales of m_1, m_2, m_3 depend on ordering scenarios. For instance, normal ordering gives $\Delta m_A^2 > 0$ while inverted ordering implies $\Delta m_A^2 < 0$ ("A" stands for atmosphere neutrinos). The mass spectrum and the orderings are independent of whether the neutrinos are of the Dirac or Majorana type [51]. However, the above derivation of neutrino oscillation is only in the quantum mechanics level and the plane wave approximation is assumed as well. A more general theory occurs in quantum field theory that includes S-matrix sandwiched in the initial and final states [30]. One can even take wave package instead of the plane wave approximation to obtain a more precise result [52, 53]. Let us next determine the transformation matrix U in Equation (2.63), and this is also the matrix combination UU^* from (2.72).

The Standard Parametrization of the PMNS Matrix

We have studied the diagonalization of the mass matrix in section 2.4, where the entries of the diagonalized mass matrix are eigenvalues in the mass eigenstate. The transformation matrix U_L transforms states between the flavor eigenstate and the mass eigenstate; e.g. see Equations (2.20 and 2.47). This matrix being an element of the Lie group can have an exponential form of expression, which leads to the fact that it can be expressed by angles and phases. For any $n \times n$ unitary matrix in a $SU(N)$ group, there are n^2 independent parameters in total. In the Dirac field case, the rotations are in the off-diagonal triangle parts and have $\frac{n(n-1)}{2}$ free parameters, and therefore the phases have $n^2 - \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$ parameters. But not all the phases are physical. The transformation from flavor to mass eigenstates in the charged current follows Equation (2.63). The PMNS matrix U is defined by $U \equiv U_L^{l\dagger} U_L^{\nu}$. It is equivalent to redefine a lepton wave function $l' = \mathrm{e}^{i\phi} l$ to cancel one column of phases in the transformation matrix, while a neutrino function $\nu' = \mathrm{e}^{i\phi}$ to extract one row of free phase from U. In this case, these phases can be chosen to be "zero" [54, 55]. The total redundant free phases in U is $2n-1$, and the physical phases are $\frac{n(n+1)}{2} - (2n-1) = \frac{(n-1)(n-2)}{2}$. For two generations, U is a 2×2 matrix, and there are $n_\theta = 1, n_\phi = 3$ angle and phase parameters, respectively, and $n_\phi^D = 0$ physical phase. For three generations, we have $n_\theta = 3, n_\phi = 6, n_\phi^P = 1$. It is explicitly shown later that the phase parameters also indicate the CP-violation. Consider the Dirac charged lepton current under CP-transformation. We can obtain the current from Equation (2.1). Take one generation in $SU(2)$ for example,

$$\mathscr{L}_{lepton} = -\frac{g_{SM}}{2} \begin{pmatrix} \overline{\nu_e} & \overline{e} \end{pmatrix}_L \gamma^\alpha \begin{pmatrix} W_\alpha^3 & W_\alpha^1 - iW_\alpha^2 \\ W_\alpha^1 + iW_\alpha^2 & -W_\alpha^3 \end{pmatrix} \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \tag{2.73}$$

In the form of three generations, we then have

$$\mathscr{L}_{lepton}^{CC} = -\frac{g_{SM}}{\sqrt{2}} \sum_i \overline{l_{iL}}(x) \gamma^\alpha U \nu_{iL}(x) W_\alpha^+ - \frac{g_{SM}}{\sqrt{2}} \sum_i \overline{\nu_{iL}}(x) \gamma^\alpha U^\dagger l_{iL}(x) W_\alpha^- \tag{2.74}$$

where $W^{\pm} = \frac{1}{\sqrt{2}}(W^1 \pm iW^2)$. According to [34, 36], the CP transformations of the Dirac lepton fields and the gauge boson fields in Dirac representation are [19, 36]

$$\begin{aligned}(CP)l_L(x)(CP)^{-1} &= \gamma^0 C\overline{l_L}^T(x') \\ (CP)\nu_L(x)(CP)^{-1} &= \gamma^0 C\overline{\nu_L}^T(x\prime) \\ (CP)W^+_\alpha(CP)^{-1} &= W^-_\alpha\end{aligned} \tag{2.75}$$

The CP-transformation of the Hermitian conjugate fields is

$$\overline{(CP)l_{iL}(x)(CP)^{-1}} = \overline{\gamma^0 C\overline{l_{iL}(x')}^T} = -l^T_{iL}(x')C^{-1}\gamma^0 \tag{2.76}$$

and it is the same for the neutrino conjugate field. Using these transformations, we are able to do the CP-transformation of Equation (2.74). Since the Lagrangian terms are singlets, we finally have

$$\begin{aligned}&(CP)\mathscr{L}^{CC}_{lepton}(CP)^{-1} \\ &= -\frac{g_{SM}}{\sqrt{2}}\overline{\nu_{iL}(x')}U^T\gamma^\alpha l_{iL}(x')W^-_\alpha - \frac{g_{SM}}{\sqrt{2}}\overline{l_{iL}(x')}(U^\dagger)^T\gamma^\alpha\nu_{iL}(x')W^+_\alpha\end{aligned} \tag{2.77}$$

where we use the property of C and gamma matrices. Compared to Equation (2.74) we find that, the charged leptonic current Lagrangian is CP invariant only if we have:

$$U = (U^\dagger)^T \qquad\qquad U^\dagger = U^T \tag{2.78}$$

this is nothing more than

$$U = U^* \tag{2.79}$$

This tells us, in the SM, if we expect that the Dirac fields are CP-invariant, the mixing matrix needs to be *Real*. Now we consider a possibly explicit three generations mixing matrix ($n = 3$). For the three independent angles, we can easily choose the rotational angle in the three orthogonal planes. Assume the initial state: $(|1\rangle, |2\rangle, |3\rangle)^T$, and we will rotate it around three axes in three steps. Step one, introduce θ_{12} around the third axis $|3\rangle$.

$$\begin{pmatrix}|1\rangle\\|2\rangle\\|3\rangle\end{pmatrix}^{(1)} = \begin{pmatrix}\cos\theta_{12} & \sin\theta_{12} & 0\\ -\sin\theta_{12} & \cos\theta_{12} & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}|1\rangle\\|2\rangle\\|3\rangle\end{pmatrix} \tag{2.80}$$

the superscript (1) stands for the "first" state that transforms after θ_{12}. Step two, add the angle θ_{13} around $|2\rangle$. Conventionally, we also add the phase shift with the rotation of θ_{13}

$$\begin{pmatrix}|1\rangle\\|2\rangle\\|3\rangle\end{pmatrix}^{(2)} = \begin{pmatrix}\cos\theta_{13} & 0 & \sin\theta_{13}\,\mathrm{e}^{-i\delta}\\ 0 & 1 & 0\\ -\sin\theta_{13}\,\mathrm{e}^{i\delta} & 0 & \cos\theta_{13}\end{pmatrix}\begin{pmatrix}|1\rangle\\|2\rangle\\|3\rangle\end{pmatrix}^{(1)} \tag{2.81}$$

In the last step, rotate θ_{23} around $|1\rangle$

$$\begin{pmatrix}|1\rangle\\|2\rangle\\|3\rangle\end{pmatrix}^{(3)} = \begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta_{23} & \sin\theta_{23}\\ 0 & -\sin\theta_{23} & \cos\theta_{23}\end{pmatrix}\begin{pmatrix}|1\rangle\\|2\rangle\\|3\rangle\end{pmatrix}^{(2)} \tag{2.82}$$

Multiply the three matrices together and we find the mixing matrix

$$\begin{aligned}U &= \begin{pmatrix}\cos\theta_{12} & \sin\theta_{12} & 0\\ -\sin\theta_{12} & \cos\theta_{12} & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos\theta_{13} & 0 & \sin\theta_{13}\,\mathrm{e}^{-i\delta}\\ 0 & 1 & 0\\ -\sin\theta_{13}\,\mathrm{e}^{i\delta} & 0 & \cos\theta_{13}\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta_{23} & \sin\theta_{23}\\ 0 & -\sin\theta_{23} & \cos\theta_{23}\end{pmatrix}\\ &= \begin{pmatrix}c_{13}c_{12} & c_{13}s_{12} & s_{13}\mathrm{e}^{-i\delta}\\ -s_{23}s_{13}c_{12}\mathrm{e}^{i\delta} - c_{23}s_{12} & -s_{23}s_{13}s_{12}\mathrm{e}^{i\delta} + c_{12}c_{23} & s_{23}c_{13}\\ -c_{23}s_{13}c_{12}\mathrm{e}^{i\delta} + s_{23}s_{12} & -c_{23}s_{13}s_{12}\mathrm{e}^{i\delta} - s_{23}c_{12} & c_{23}c_{13}\end{pmatrix}\end{aligned} \tag{2.83}$$

where we redefine $\sin\theta_{12} \equiv s_{12}, \cos\theta_{13} \equiv c_{13}, ...$ for simpler notation. This is so-called the "the standard parametrization" [34]. From the discussion above we notice that, the CP conservation for the Dirac fields will leave the phase shift zero $\delta = 0$. In the other hand, it is a way of finding the CP-violation by observing the non-trivial phase shift in mixing of Dirac particle states.

It is however a bit different in the Majorana case. Remember that the Majorana field in the SM fixing is not gauge invariant, which means that the neutrino fields cannot be performed by a rephasing, i.e. the term $\overline{\nu_L^c}\nu_L + h.c.$ will change (after the rephasing). Thus, the physical independent phases are in total $n_\phi^M = \frac{n(n+1)}{2} - n = \frac{n(n-1)}{2} = n_\theta$. In the two generations case, we have $n_\theta = 1, n_\phi^M = 1$, while there is $n_\theta = 3, n_\phi^M = 3$ for three generations. Since the Majorana particles are their own antiparticles, we should have the equality

$$(CP)\nu_{iL}(x)(CP)^{-1} = (CP)\nu_{iL}^c(CP)^{-1} \tag{2.84}$$

and this gives

$$\begin{aligned}&\eta^*\gamma^0 C\overline{\nu_{iL}}^T(x') = \eta C(\gamma^0)^T C^{-1} C\overline{\nu_{iL}}^T(x')\\ \Rightarrow\quad &\eta^*\gamma^0 C\overline{\nu_{iL}}^T(x') = -\eta\gamma^0 C\overline{\nu_{iL}}^T(x')\end{aligned} \tag{2.85}$$

where we use $C(\gamma^0)^T C^{-1} = -\gamma^0$. And this implies that

$$\eta^* = -\eta \qquad\qquad \Rightarrow \qquad\qquad \eta = \pm i \tag{2.86}$$

where η is the CP transformation phase space factor value. This will end in $U^* = \pm U$ in the CP transformation of charged current, which means the mixing matrix is pure Imaginary/Real if CP conserves. Thus, it is straightforward for us to write a Majorana mixing matrix [54]

$$U^M = US^M \qquad\qquad S^M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{e}^{i\delta_1} & 0 \\ 0 & 0 & \mathrm{e}^{i(\delta_2+\delta)} \end{pmatrix} \tag{2.87}$$

There are three phase shifts $\delta, \delta_1, \delta_2$. In fact, one can randomly create the Majorana mixing matrix if only all the entries include at least a phase shift. To satisfy CP-invariance, we have

$$\mathrm{e}^{2i\delta_{1,2}} = \pm 1 \qquad\qquad \textit{and} \qquad\qquad \mathrm{e}^{\pm 2i\delta} = \pm 1 \tag{2.88}$$

3 The Minimal Left-Right Symmetric Model

3.1 Basics of the Model: Why "Left-Right"

When we say "the Standard Model", we are talking about the Yang-Mills gauge theory for quark-gluon fields which is precisely a $SU(3)$ symmetry for colors $SU(3)_C$, combined with the electroweak interactions of the left-handed $SU(2)_L \times U(1)_Y$ symmetry which we have discussed in the last chapter. Namely, $SU(3)_C \times SU(2)_L \times U(1)_Y$ in total. Strictly speaking, the mass generated by the right-handed singlet neutrino is also "beyond" the SM. In the SM we only have massless neutrino and right-handed singlet charged lepton. However, "beyond" is accurately said to mean that the symmetry group is larger than the SM's. For those singlets introduced in the SM we prefer to call them the fixing of the SM.

A natural way of thinking the BSM is to find higher symmetry which includes the SM physics and can reduce to the SM in specific circumstances [56, 57]. The smallest symmetry in this way is the mLRSM (minimal left-right symmetric model), $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_X$. This model has been studied for more than 30 years, and there are many higher symmetry versions. But they have the same basic idea: to create a parity restored model that will be violated by spontaneous mechanism.

We have mentioned in this simplest BSM that the parity is restored at first and violated when the model spontaneously breaks into the SM. This could explain the universal parity asymmetry in the SM, which is one of the reasons people build mLRSM [58, 59]. The Majorana neutrinos are not generated in the singlets but in a gauge invariant way, which makes the observation possible in this scenario. This also gives constraints on the mass scale of heavy neutrinos and also offers explanations for the hierarchy problem between light neutrinos and charged leptons, $m_{\nu_l} \ll m_l$ [60]. The mLRSM in this way is a simple, natural extension of the SM and will reduce to the SM through spontaneous symmetry breaking by a large right-handed VEV. The rough graph of the process is

$$\begin{aligned} & SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_X \\ \xrightarrow{\langle \Delta_R \rangle} \; & \\ & SU(3)_C \times SU(2)_L \times U(1)_Y \\ \xrightarrow{\langle \mathbf{\Phi} \rangle} \; & \\ & SU(3)_C \times U(1)_{EM} \end{aligned} \tag{3.1}$$

We begin from the Lagrangian of the mLRSM. Consider the minimal symmetry group as $SU(2)_L \times SU(2)_R \times U(1)_X$, where $X = B - L$ is baryon number minus lepton number, which is conserved in this model [61]. Same as in the SM, the $SU(3)_C$ group for color fields under strong interaction is not taken into account. In this way, we have seven gauge fields, three W_L^a and three W_R^a for left- and right-handed $SU(2)_{L,R}$ and B_μ for $U(1)_X$. The superscript a stands for the generators of the $\mathfrak{su}(2)$ lie algebra. In this larger symmetry group, the right-handed neutrinos are introduced to form the correspondent right-handed doublet with the existing right-handed charged leptons. The left- and right-handed fermions are considered to have the same symmetry group in the LRSM. This naturally leads to the fact that the parity is conserved in the LRSM. In one of the processes, the parity is violated spontaneously through the Higgs mechanism due to the unequal VEVs of the left- and right-handed Higgs fields (in our case are Higgs triplets) [57, 58, 59]. Readers will find the explicit calculations of Higgs mass and gauge boson mass in the subsequent sections. The Lagrangian of the model should contain both Dirac terms and Majorana terms for fermions in order to give es in the seesaw mechanism. We can then write the full Lagrangian without the strong interaction terms of the gluon as [60, 62],

$$\mathscr{L} = \mathscr{L}_{\mathrm{g}} + \mathscr{L}_{\mathrm{f}} + \mathscr{L}_H + \mathscr{L}_Y + V(\phi, \tilde{\phi}, \Delta_L, \Delta_R) \tag{3.2}$$

where "g", "f", "H", and "Y" represent the gauge, fermions, Higgs, and Yukawa under the mLRSM, respectively. We have used the same subscript as in the SM, but each terms are definitely not equivalent. Let us define each Lagrangian in (3.2). The bosonic gauged Lagrangian is defined as

$$\mathscr{L}_{\mathrm{g}} = -\frac{1}{4} W_L^{\mu\nu,a} W_{L,\mu\nu}^a - \frac{1}{4} W_R^{\mu\nu,a} W_{R,\mu\nu}^a - \frac{1}{4} B^{\mu\nu} B_{\mu\nu} \tag{3.3}$$

where the field strength tensor has the same expressions as in the SM, $W_{L,R}^{\mu\nu,a} = \partial^\mu W_{L,R}^{\nu,a} - \partial^\nu W_{L,R}^{\mu,a}$, $B^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu$. The fermion kinetic term with gauge fixing is defined as,

$$\begin{aligned} \mathscr{L}_{\mathrm{f}} = \sum_{\Psi = Q, L} \big\{ & \bar{\Psi}_L i\gamma^\mu (\partial_\mu + i g_L \frac{\vec{\sigma}}{2} \cdot \vec{W}_{L,\mu} + i g' \frac{X}{2} B_\mu) \Psi_L \\ & + \bar{\Psi}_R i\gamma^\mu (\partial_\mu + i g_R \frac{\vec{\sigma}}{2} \cdot \vec{W}_{R,\mu} + i g' \frac{X}{2} B_\mu) \Psi_R \big\} \end{aligned} \tag{3.4}$$

The Higgs kinetic term with gauge fixing is defined as

$$\mathscr{L}_H = \mathrm{Tr}\big[(D_\mu \phi)^\dagger (D^\mu \phi) + (D_\mu \Delta_L)^\dagger (D^\mu \Delta_L) + (D_\mu \Delta_R)^\dagger (D^\mu \Delta_R)\big] \tag{3.5}$$

Field	$SU(3)_C$	$SU(2)_L$	$SU(2)_R$	$U(1)_X$
Q_L	3	2	1	$\frac{1}{3}$
Q_R	3	1	2	$\frac{1}{3}$
L_L	1	2	1	-1
L_R	1	1	2	-1
ϕ	1	2	2	0
$\tilde{\phi}$	1	2^*	2^*	0
Δ_L	1	3	1	2
Δ_R	1	1	3	2

Table 3.1: fundamental representation Fields degrees in mLRSM

the Yukawa term for both Dirac type and Majorana type is defined as,

$$
\begin{aligned}
\mathscr{L}_Y = &- \sum_{\Psi=Q,L} \left\{ \bar{\Psi}_{Li} \Gamma^{\Psi}_{ij} \phi \Psi_{Rj} + \bar{\Psi}_{Li} \tilde{\Gamma}^{\Psi}_{ij} \tilde{\phi} \Psi_{Rj} + h.c. \right\} \\
&- \sum_{Leptons} \left\{ \overline{L^c_{Li}} G_{L,ij} i\sigma_2 \Delta_L L_{Lj} + \overline{L^c_{Ri}} G_{R,ij} i\sigma_2 \Delta_L L_{Rj} + h.c. \right\}
\end{aligned}
\tag{3.6}
$$

where $\tilde{\phi} = \sigma_2 \phi^* \sigma_2$. $\vec{\sigma}$ are the generators of the $\mathfrak{su}(2)$ algebra, i.e. Pauli matrices in fundamental representation. $\tilde{\phi}$ is a gauge fixing field to ensure gauge invariance. In $SU(2)$, ϕ and $\tilde{\phi}$ are bidoublets, and $\Delta_{L,R}$ are triplets of the scalar fields, in order to contribute in fermions masses. The sum over Q and L is the quark and lepton doublet on the basis of the flavor eigenstates,

$$
Q_L = \begin{pmatrix} u \\ d \end{pmatrix}_L \equiv [3, 2, 1, \frac{1}{3}] \qquad L_L = \begin{pmatrix} v_l \\ l \end{pmatrix}_L \equiv [1, 2, 1, -1] \tag{3.7}
$$

$$
Q_R = \begin{pmatrix} u \\ d \end{pmatrix}_R \equiv [3, 1, 2, \frac{1}{3}] \qquad L_R = \begin{pmatrix} v_l \\ l \end{pmatrix}_R \equiv [1, 1, 2, -1] \tag{3.8}
$$

where u and d are up types and down types quarks, respectively. We think about three generations of quarks and leptons. $l = e, \mu, \tau$. The numbers in the square brackets represent the correspond symmetries in the mLRSM group. For example, in the fundamental representation (or spinor representation), this number is $2s + 1$ the degree of the group where s is the correspond spin (quarks have a color field of 3 degrees, the correspondent of the spin-1 gauge field).

Physically, these numbers in Table 3.1 show whether the fields take part in the interaction of the symmetry group or not. Left quarks take place in the color field and the left-handed electroweak field, but do not have any degree in the right-handed field. In other words, they participate in the strong, weak, and electromagnetic interactions. And the last number gives the magnitude of the interaction in the symmetry group. It is the conserved charge number, and in mLRSM it is $X = B - L$

coincidentally. L^c in Lagrangian $\mathscr{L}_Y$ is the particle-antiparticle conjugate field of L and the definition is in Equation (2.25). Its Lorentz invariant conjugate $\overline{L^c}$ is defined in Equation (2.38). $V(\phi, \tilde{\phi}, \Delta_L, \Delta_R)$ is the possible potential. Γ^{Ψ}_{ij}, Δ^{Ψ}_{ij}, $G_{L,R,ij}$ are matrices for flavor mixing (the dummy subscripts here represent the flavor states). C^{-1} is the inverse of the charge conjugate operator. We will start from the interaction potential to calculate the VEVs.

3.2 Higgs Potential

We have mentioned that for any symmetry group larger than $O(1)$, there could be spontaneous symmetry breaking (SSB) that generates Goldstone bosons through non-trivial VEVs of the scalar fields. In the SM, there is a Higgs scalar field which has non-zero VEV, that breaks the electroweak symmetry $SU(2)_L \times U(1)_Y$ to electromagnetic symmetry $U(1)_{EM}$, where the three gauge bosons W, Z "eat" the Goldstone bosons to gain masses and leave the fourth gauge field massless which is the photon field [21]. The first step we need to take is to calculate the VEVs. Similarly in the ϕ^4 toy model (see section 2.1), the VEVs can be obtained by solving equations of motions with respect to the Higgs fields $\Delta_{L,R}$ and ϕ,

$$\frac{\partial V(\phi, \tilde{\phi}, \Delta_L, \Delta_R)}{\partial \Delta_L} = \frac{\partial V(\phi, \tilde{\phi}, \Delta_L, \Delta_R)}{\partial \Delta_R} = 0; \qquad \frac{\partial V(\phi, \tilde{\phi}, \Delta_L, \Delta_R)}{\partial \phi} = 0 \tag{3.9}$$

By doing this, we first need to write down the explicit form of the potential, which should include all possible Hermitian combinations of the four Higgs fields and its interactions. The result is the following [60, 63],

$$\begin{aligned}
&V(\phi, \tilde{\phi}, \Delta_L, \Delta_R) \\
= &-\sum_{i,j}^{2} \mu_{ij}^2 \mathrm{Tr}[\phi_i^\dagger \phi_j] + \sum_{i,j,k,l}^{2} \lambda_{ijkl} \mathrm{Tr}[\phi_i^\dagger \phi_j] \mathrm{Tr}[\phi_k^\dagger \phi_l] + \sum_{i,j,k,l}^{2} \lambda'_{ijkl} \mathrm{Tr}[\phi_i^\dagger \phi_j \phi_k^\dagger \phi_l] \\
&- \mu^2 \mathrm{Tr}[\Delta_L^\dagger \Delta_L + \Delta_R^\dagger \Delta_R] + \rho_1 \big\{ (\mathrm{Tr}[\Delta_L^\dagger \Delta_L])^2 + (\mathrm{Tr}[\Delta_R^\dagger \dagger_R])^2 \big\} \\
&+ \rho_2 \big\{ \mathrm{Tr}[\Delta_L^\dagger \Delta_L \Delta_L^\dagger \Delta_L] + \mathrm{Tr}[\Delta_R^\dagger \Delta_R \Delta_R^\dagger \Delta_R] \big\} + \rho_3 \mathrm{Tr}[\Delta_L^\dagger \Delta_L \Delta_R^\dagger \Delta_R] \\
&+ \sum_{i,j}^{2} \alpha_{i,j} \mathrm{Tr}[\phi_i^\dagger \phi_j] (\mathrm{Tr}[\Delta_L^\dagger \Delta_L] + \mathrm{Tr}[\Delta_R^\dagger \Delta_R]) \\
&+ \sum_{i,j}^{2} \beta_{i,j} (\mathrm{Tr}[\Delta_L^\dagger \Delta_L \phi_i \phi_j^\dagger] + \mathrm{Tr}[\Delta_R^\dagger \Delta_R \phi_i^\dagger \phi_j]) + \sum_{i,j}^{2} \gamma_{i,j} \mathrm{Tr}[\Delta_L^\dagger \phi_i \Delta_R \phi_j^\dagger]
\end{aligned} \tag{3.10}$$

where the summation indices $1, 2$ stands for summing over the Higgs fields $\phi_1 \equiv \phi$, $\phi_2 \equiv \tilde{\phi}$. ϕ, Δ_L and Δ_R represent Higgs bosons. We add a more explicit version of the potential that shows every term in the summation in the Appendix A.

Higgs triplets are defined to couple with the Majorana terms of lepton doublets.

The only possible non-zero VEVs come from the neutral components of the matrices. The left-handed Higgs triplet can generally be defined as

$$\Delta_L = \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix} \tag{3.11}$$

In the mLRSM, the total charge also contains the third component of the right-handed $SU(2)_R$ isospin. Now acting this charge operator $\hat{Q} = I_{L3} + I_{R3} + \frac{1}{2}\hat{X}$ on the triplet and the vacuum, we would have

$$\begin{aligned}
\hat{Q}\Delta_L |0\rangle &= [\hat{Q}, \Delta_L] |0\rangle + \Delta_L \hat{Q} |0\rangle = [I_{L3}, \Delta_L] |0\rangle + \Delta_L \hat{Q} |0\rangle \\
&= \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix} - \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] |0\rangle + 2 \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix} |0\rangle \\
&= 2 \begin{pmatrix} \Delta_{11} & 2\Delta_{12} \\ 0 & \Delta_{22} \end{pmatrix} |0\rangle
\end{aligned} \tag{3.12}$$

where we can see that there is only one zero entry, which could be a possibility for the non-trivial VEV. Thereby, the VEVs of the Higgs triplets can be

$$\langle \Delta_L \rangle = \begin{pmatrix} 0 & 0 \\ \nu_L & 0 \end{pmatrix} \qquad \langle \Delta_R \rangle = \begin{pmatrix} 0 & 0 \\ \nu_R & 0 \end{pmatrix} \tag{3.13}$$

Acting the operator to the Higgs bidoublet, we obtain the VEVs as well.

$$\begin{aligned}
\langle \phi \rangle &= \begin{pmatrix} \kappa & 0 \\ 0 & \kappa' \end{pmatrix} \\
\langle \tilde{\phi} \rangle &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \kappa^* & 0 \\ 0 & \kappa'^* \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} \kappa'^* & 0 \\ 0 & \kappa^* \end{pmatrix}
\end{aligned} \tag{3.14}$$

Now we insert all the VEVs into the potential (3.10) and use the shorthand writing (A.2) to calculate the trace. After some algebras, we have

$$\begin{aligned}
&V(\langle \phi \rangle, \langle \tilde{\phi} \rangle, \langle \Delta_L \rangle, \langle \Delta_R \rangle) \\
=& -\mu_1^2(|\kappa|^2 + |\kappa'|^2) - 2\mu_2^2(\kappa^*\kappa'^* + \kappa'\kappa) + \lambda_1(|\kappa|^2 + |\kappa'|^2) \\
&+ 2\lambda_2(\kappa^*\kappa'^* + \kappa'\kappa)(|\kappa|^2 + |\kappa'|^2) + \lambda_3[(\kappa^*\kappa'^* + \kappa'^*\kappa^*)^2 + (\kappa'\kappa + \kappa\kappa')^2] \\
&+ \lambda_4(\kappa^*\kappa'^* + \kappa'^*\kappa^*)(\kappa'\kappa + \kappa\kappa') + \lambda_1'(|\kappa|^4 + |\kappa'|^4) + \lambda_2'(\kappa^*\kappa'^* + \kappa'\kappa)(|\kappa|^2 + |\kappa'|^2) \\
&+ 2\lambda_3'|\kappa|^2|\kappa'|^2 - \mu^2(|\nu_L|^2 + |\nu_R|^2) + (\rho_1 + \rho_2)(|\nu_L|^4 + |\nu_R|^4) + \rho_3|\nu_L|^2|\nu_R|^2 \\
&+ (\alpha_1 + \alpha_3)(|\kappa|^2 + |\kappa'|^2)(|\nu_L|^2 + |\nu_R|^2) + 2\alpha_2(\kappa^*\kappa'^* + \kappa'\kappa)(|\nu_L|^2 + |\nu_R|^2) \\
&+ \beta_1|\kappa|^2(|\nu_L|^2 + |\nu_R|^2) + \beta_2(|\nu_L|^2 + |\nu_R|^2)(\kappa\kappa' + \kappa^*\kappa' *) + \beta_3|\kappa'|^2(|\nu_L|^2 + |\nu_R|^2) \\
&+ \gamma_1(\nu_L^*\nu_R + \nu_R^*\nu_L)\kappa\kappa^* + \gamma_2(\nu_L^*\nu_R\kappa\kappa' + \nu_L^*\nu_R\kappa^*\kappa'^*) + \gamma_3(\nu_L^*\nu_R + \nu_R^*\nu_L)\kappa'^*\kappa'
\end{aligned} \tag{3.15}$$

For simplification, we assume that VEVs κ, κ', ν_L, and ν_R are real numbers. This means that we neglect the phase shift between VEVs or assume that they have the same phase. After making this simplification, we obtain

$$\begin{aligned} V(\kappa, \kappa', \nu_L, \nu_R) = & -\mu_1^2(\kappa^2 + \kappa'^2) - 4\mu_2^2\kappa\kappa' \\ & + \lambda_1(\kappa^2 + \kappa'^2)^2 + (4\lambda_2 + 2\lambda_2')(\kappa^2 + \kappa'^2) \\ & + (8\lambda_3 + 4\lambda_4 + 2\lambda_3')\kappa^2\kappa'^2 + \lambda_1'(\kappa^4 + \kappa'^4) \\ & - \mu^2(\nu_L^2 + \nu_R^2) + (\rho_1 + \rho_2)(\nu_L^4 + \nu_R^4) + \rho_3\nu_L^2\nu_R^2 \\ & + [(\alpha_1 + \alpha_3)(\kappa^2 + \kappa'^2) + 4\alpha_2\kappa\kappa'](\nu_L^2 + \nu_R^2) \\ & + (\beta_1\kappa^2 + 2\beta_2\kappa\kappa' + \beta_3\kappa'^2])(\nu_L^2 + \nu_R^2) \\ & + 2(\gamma_1\kappa^2 + \gamma_2\kappa\kappa' + \gamma_3\kappa'^2)\nu_L\nu_R \end{aligned} \tag{3.16}$$

The next step is to find the fields conditions at the global minimum. Put the VEVs in the Equation (3.9). The first derivatives with respect to the VEVs of the Higgs triplet are the following, where we suppose $\nu_{L,R} \neq 0$.

$$\begin{aligned} \frac{\partial V}{\partial \nu_L} = & 2\mu_2\nu_L + 4(\rho_1 + \rho_2)\nu_L^3 + 2[\rho_3\nu_R^2\nu_L + 2(\alpha_1 + \alpha_3)(\kappa^2 + \kappa'^2) + 4\alpha_2\kappa\kappa']\nu_L \\ & + 2[\beta_1\kappa^2 + 2\beta_2\kappa\kappa' + \beta_3\kappa'^2]\nu_L + 2(\gamma_1\kappa^2 + \gamma_2\kappa\kappa' + \gamma_3\kappa'^2)\nu_R = 0 \end{aligned} \tag{3.17}$$

$$\begin{aligned} \frac{\partial V}{\partial \nu_R} = & 2\mu_2\nu_R + 4(\rho_1 + \rho_2)\nu_L^3 + 2[\rho_3\nu_L^2\nu_R + 2(\alpha_1 + \alpha_3)(\kappa^2 + \kappa'^2) + 4\alpha_2\kappa\kappa']\nu_L \\ & + 2[\beta_1\kappa^2 + 2\beta_2\kappa\kappa' + \beta_3\kappa'^2]\nu_R + 2(\gamma_1\kappa^2 + \gamma_2\kappa\kappa' + \gamma_3\kappa'^2)\nu_L = 0 \end{aligned} \tag{3.18}$$

Also we need to check if the second derivatives are positive in order to reach the minimal potential. Multiply the first and the second equations with ν_R and ν_L respectively, and subtract them. After doing some algebra, we finally arrive at either one of the constraints of the VEVs [64],

$$\textbf{(a)} \quad {\nu_L}^2 = {\nu_R}^2 \qquad\qquad or \tag{3.19}$$

$$\textbf{(b)} \quad \nu_L\nu_R = \frac{2(\gamma_1\kappa^2 + \gamma_2\kappa\kappa' + \gamma_3\kappa'^2)}{4(\rho_1 + \rho_2) - 2\rho_3} \tag{3.20}$$

The first constraint is a trivial one, and the parity does not violate after the SSB of the left-right symmetry. We will discuss the constraints of mass scale and the VEVs scale after finishing the calculation of $W_{L,R}$ mixing from the Higgs mechanism of gauge fields interacting with leptons. We can also do the same procedure for κ and κ', and the constraint is:

$$\textbf{(c)} \quad \kappa\kappa' = \frac{(4\alpha_2 + 2\beta_2)(\nu_L^2 + \nu_R^2) + 2\gamma_2\nu_L\nu_R}{4\gamma_1'} \tag{3.21}$$

In fact, apart from the neutral fields which can generate the non-trivial VEVs, there are Higgs scalar charged fields as well. Using the charge operator eigenfunction (3.12), we can define the general Higgs triplet assignments as

$$\Delta = \begin{pmatrix} \frac{1}{\sqrt{2}}\delta^+ & \delta^{++} \\ \delta^0 & -\frac{1}{\sqrt{2}}\delta^+ \end{pmatrix} \tag{3.22}$$

where we consider the $SU(2)$ triplet

$$\Delta = \frac{1}{\sqrt{2}}\tau_\alpha\delta_\alpha = \frac{1}{\sqrt{2}}\begin{pmatrix} \delta_3 & \delta_1 - i\delta_2 \\ \delta_1 - i\delta_2 & -\delta_3 \end{pmatrix} \tag{3.23}$$

One can also use minus sign in the superscript, since positive and negative fields have the same mathematical features. The Higgs potential (3.10) gives the self interaction of Higgs fields. If the CP invariance is taken into account, all the β, γ coefficients in (3.10) will be set to zero. This also provides an advantage that less number of parameters can keep simple for the fine tuning or even remove it [65]. In order to discuss the scalar propagator, it is necessary to calculate the mass scale of the left- and right-handed charged Higgs fields. The masses of the charged Higgs fields come from the terms that contain Δ_L, Δ_R in (3.10). These terms are

$$\begin{aligned}
&-\mu^2\mathrm{Tr}[\Delta_L^\dagger\Delta_L + \Delta_R^\dagger\Delta_R] + \rho_1\big\{(\mathrm{Tr}[\Delta_L^\dagger\Delta_L])^2 + (\mathrm{Tr}[\Delta_R^\dagger\Delta_R])^2\big\} \\
&+ \rho_2\big\{\mathrm{Tr}[\Delta_L\Delta_L]\mathrm{Tr}[\Delta_L^\dagger\Delta_L^\dagger] + \mathrm{Tr}[\Delta_R\Delta_R]\mathrm{Tr}[\Delta_R^\dagger\Delta_R^\dagger]\big\} + \rho_3\mathrm{Tr}[\Delta_L^\dagger\Delta_L]\mathrm{Tr}[\Delta_R^\dagger\Delta_R] \\
&+ \rho_4\big\{\mathrm{Tr}[\Delta_L\Delta_L]\mathrm{Tr}[\Delta_R^\dagger\Delta_R^\dagger] + \mathrm{Tr}[\Delta_R\Delta_R]\mathrm{Tr}[\Delta_L^\dagger\Delta_L^\dagger]\big\} \\
&+ \alpha_1\mathrm{Tr}[\phi^\dagger\phi](\mathrm{Tr}[\Delta_L^\dagger\Delta_L] + \mathrm{Tr}[\Delta_R^\dagger\Delta_R]) \\
&+ \alpha_3\mathrm{Tr}[\tilde{\phi}^\dagger\tilde{\phi}](\mathrm{Tr}[\Delta_L^\dagger\Delta_L] + \mathrm{Tr}[\Delta_R^\dagger\Delta_R]) \\
&+ \alpha_2\mathrm{Tr}[\phi^\dagger\tilde{\phi} + \tilde{\phi}^\dagger\phi](\mathrm{Tr}[\Delta_L^\dagger\Delta_L] + \mathrm{Tr}[\Delta_R^\dagger\Delta_R])
\end{aligned} \tag{3.24}$$

We insert the VEVs of the bidoublet Higgs fields directly, and for mass discussion we only keep the quadratic terms of $\delta_{L,R}^{++}$ with the Higgs triplet VEVs $\delta_{L,R}^0$. Suppose the fields and the VEVs are all real, and satisfy the approximation $\nu_R \gg \nu_L$ and $\nu_R \gg \kappa, \nu_R \gg \kappa'$, we obtain

$$\begin{aligned}
=&-\mu^2(\delta_L^{++2} + \delta_R^{++2}) + 2(\rho_1 + 2\rho_2)({\delta_L^0}^2\delta_L^{++2} + {\delta_R^0}^2\delta_R^{++2}) + \rho_3({\delta_L^0}^2\delta_R^{++2} + {\delta_R^0}^2\delta_L^{++2}) \\
&+ 8\rho_4\delta_L^0\delta_R^0\delta_L^{++2}\delta_R^{++2} + (\alpha_1 + \alpha_3)(\kappa^2 + \kappa'^2)(\delta_L^{++2} + \delta_R^{++2}) + 4\alpha_2\kappa\kappa'(\delta_L^{++2} + \delta_R^{++2}) \\
=&-\mu^2(\delta_L^{++2} + \delta_R^{++2}) + \begin{pmatrix}\delta_L^{++} & \delta_R^{++}\end{pmatrix}\begin{pmatrix} \rho_3{\delta_R^0}^2 + \kappa_0 & 4\rho_4\delta_L^0\delta_R^0 \\ 4\rho_4\delta_L^0\delta_R^0 & 2(\rho_1 + 2\rho_2){\delta_R^0}^2 + \kappa_0 \end{pmatrix}\begin{pmatrix}\delta_L^{++} \\ \delta_R^{++}\end{pmatrix}
\end{aligned} \tag{3.25}$$

The superscript 2 stands for square. In the last line, the terms with ${\delta_L^0}^2$ have been neglected since δ_L^0 refers to the left-handed Higgs VEV ν_L. Redefine a simpler shorthand $\kappa_0 \equiv (\alpha_1 + \alpha_3)(\kappa^2 + \kappa'^2) + 4\alpha_2\kappa\kappa'$. The mass of the charged Higgs fields are

$$m^2_{\delta_L^{\pm\pm}} = -\mu^2 + \frac{1}{2}(\rho_3\nu_R^2 + 2(\rho_1 + 2\rho_2)\nu_R^2 + 2\kappa_0) - \frac{1}{2}\sqrt{\mathbf{\Delta}} \tag{3.26}$$
$$m^2_{\delta_R^{\pm\pm}} = -\mu^2 + \frac{1}{2}(\rho_3\nu_R^2 + 2(\rho_1 + 2\rho_2)\nu_R^2 + 2\kappa_0) + \frac{1}{2}\sqrt{\mathbf{\Delta}} \tag{3.27}$$

where $\mathbf{\Delta}$ is the discriminant,

$$\begin{aligned}\mathbf{\Delta} =&(\rho_3\nu_R^2 + 2(\rho_1 + 2\rho_2)\nu_R^2 + 2\kappa_0)^2 \\ &- 4[(\rho_3\nu_R^2 + \kappa_0)(2(\rho_1 + 2\rho_2)\nu_R^2 + \kappa_0) - 16\rho_4^2\nu_L\nu_R]\end{aligned} \tag{3.28}$$

the square root can be reduced due to the approximation, that is

$$\sqrt{\mathbf{\Delta}} \approx \rho_3\nu_R^2 - 2(\rho_1 + 2\rho_2)\nu_R^2 \tag{3.29}$$

put this back and finally we obtain the mass scale

$$m^2_{\delta_L^{\pm\pm}} = -\mu^2 + 2(\rho_1 + 2\rho_2)\nu_R^2 + 2\kappa_0 \tag{3.30}$$
$$m^2_{\delta_R^{\pm\pm}} = -\mu^2 + \rho_3\nu_R^2 + 2\kappa_0 \tag{3.31}$$

Since the kinetic mass (if without the potential) is $-\mu^2 > 0$, the mass scale of both left- and right-handed double-charged Higgs fields are at least ν_R, i.e. the mass scale of both fields are very large. The procedures for searching for masses of Higgs scalars are identical. We can also figure out the mass expressions for Higgs bidoublet with respect to coefficients and VEVs [64, 65]. The crucial aspects of the calculation are the approximations and the arguments such as "the CP-invariance" [66].

3.3 Higgs Kinetic Terms

Let us do Higgs mechanism very carefully term by term and see explicitly how the gauge bosons $W^\mu_{L,R}$ and B_μ "eat" the Goldstone bosons and gain masses in mLRSM. Practically, we can write down the explicit forms of the covariance derivative in the Lagrangian (3.2) [62]

$$D_\mu\phi = \partial_\mu\phi + ig(\frac{\vec{\sigma}}{2}\cdot\vec{W}_L\phi - \phi\frac{\vec{\sigma}}{2}\cdot\vec{W}_R) \tag{3.32}$$
$$D_\mu\Delta_L = \partial_\mu\Delta_L + ig[\frac{\vec{\sigma}}{2}\cdot\vec{W}_L,\ \Delta_L] + ig'\frac{X}{2}B_\mu\Delta_L \tag{3.33}$$
$$D_\mu\Delta_R = \partial_\mu\Delta_R + ig[\frac{\vec{\sigma}}{2}\cdot\vec{W}_R,\ \Delta_R] + ig'\frac{X}{2}B_\mu\Delta_R \tag{3.34}$$

where we assume that the chirality symmetry is restored in this model: $g_L = g_R \equiv g$. In Equation (3.32), one can choose "Left-Right" or "Right-Left" ("dash" stands for minus sign). This will not make a difference to the result, since we are interested in the trace. We can see this from the explicit derivation of the first bosonic kinetic term in the Appendix B.

The gauge-scalar boson interaction expressions are derived from the kinetic term of the Lagrangian. In the calculating of the neutrinoless double decay rate, we need the coupling of the Higgs triplet-gauge boson interaction which can be obtained by inserting the Higgs field (3.22) into the trace,

$$\mathrm{Tr}[(D_\mu\Delta_L)^\dagger(D^\mu\Delta_L)] + \mathrm{Tr}[(D_\mu\Delta_R)^\dagger(D^\mu\Delta_R)] \tag{3.35}$$

After doing calculations (for explicit derivations, see Appendix B), we arrive at the interaction,

$$-g^2(W^+_{L\mu}W^{+,\mu}_L + W^-_{L\mu}W^{-,\mu}_L)\delta^0_L\delta^{++}_L - g^2(W^+_{R\mu}W^{+,\mu}_R + W^-_{R\mu}W^{-,\mu}_R)\delta^0_R\delta^{++}_R \tag{3.36}$$

where $W^\pm = \frac{1}{\sqrt{2}}(W^1 \pm W^2)$. The coupling of the W-W-H^{++} vertex is proportional to $-g^2\delta^0_{L/R}$. After the mass mechanism from inserting the VEVs, the coupling is $-g^2\nu_{L/R}$, where $\nu_{L/R}$ is the VEVs of left- or right-handed Higgs triplets. In some literature [65, 67], the VEVs in (3.13) are defined as $\frac{\nu_{L/R}}{\sqrt{2}}$, and the couplings are $-\frac{g^2}{\sqrt{2}}\nu_{L/R}$. In the next step we insert the VEVs and decouple the mixing fields to obtain the mass scale of the gauge bosons. The traces of the kinetic terms finally become

$$\begin{aligned}
&\mathrm{Tr}\big[(D_\mu\phi)^\dagger(D^\mu\phi) + (D_\mu\Delta_L)^\dagger(D^\mu\Delta_L) + (D_\mu\Delta_R)^\dagger(D^\mu\Delta_R)\big] \\
= \;& \frac{g^2}{2}(\kappa^2 + \kappa'^2 + 2\nu_L^2)W^+_{L,\mu}W^{-,\mu}_L + \frac{g^2}{2}(\kappa^2 + \kappa'^2 + 2\nu_R^2)W^+_{R,\mu}W^{-,\mu}_R \\
&+ \frac{g^2}{4}(\kappa^2 + \kappa'^2 + 4\nu_L^2)W^2_{3L} + \frac{g^2}{4}(\kappa^2 + \kappa'^2 + 4\nu_R^2)W^2_{3R} \\
&- \frac{g^2}{2}(\kappa^2 + \kappa'^2)W_{3L}W_{3R} - g^2\kappa\kappa' W^-_{L,\mu}W^{+,\mu}_R - g^2\kappa\kappa' W^-_{R,\mu}W^{+,\mu}_L \\
&- 2gg'(\nu_L^2 W_{3L,\mu}B_\mu + \nu_R^2 W_{3R,\mu}B^\mu) + g'^2(\nu_L^2 + \nu_R^2)B_\mu B^\mu
\end{aligned} \tag{3.37}$$

The explicit calculation is found in the Appendix B. The mass scales of W_L and W_R are proportional to ν_L^2 and ν_R^2 respectively. In the last line of (3.37), we enter the hypercharge number $X[\Delta_{L,R}] = 2$ that belongs to the $U(1)_X$ symmetry from table 3.1. One can directly take the hypercharge number in the first place in (3.34). However, the fields W_μ and B_μ in (3.37) are coupled with each other. They are not pure mass terms.

3.3.1 Masses of the Gauge Bosons

We need to decouple the interactions, i.e., to transform the gauge boson states to the mass eigenstates. We have W^+, W^- mixing terms and W_{3L}, W_{3R}, B mixing terms, so there are more than one transformation matrices. We start from the charged gauge bosons $W^{+,-}_{L,R}$.

$$
\begin{aligned}
&\frac{g^2}{2}(\kappa^2+\kappa'^2+2\nu_L^2)W^+_{L,\mu}W_L^{-,\mu}+\frac{g^2}{2}(\kappa^2+\kappa'^2+2\nu_R^2)W^+_{R,\mu}W_R^{-,\mu}\\
&-g^2\kappa\kappa' W^-_{L,\mu}W_R^{+,\mu}-g^2\kappa\kappa' W^-_{R,\mu}W_L^{+,\mu}\\
=&\begin{pmatrix}W_L^+ & W_R^-\end{pmatrix}\begin{pmatrix}\frac{g^2}{4}(\kappa^2+\kappa'^2+2\nu_L^2) & -\frac{1}{2}g^2\kappa\kappa'\\ -\frac{1}{2}g^2\kappa\kappa' & \frac{g^2}{4}(\kappa^2+\kappa'^2+2\nu_R^2)\end{pmatrix}\begin{pmatrix}W_L^+\\ W_R^-\end{pmatrix}\\
&+\begin{pmatrix}W_L^- & W_R^+\end{pmatrix}\begin{pmatrix}\frac{g^2}{4}(\kappa^2+\kappa'^2+2\nu_L^2) & -\frac{1}{2}g^2\kappa\kappa'\\ -\frac{1}{2}g^2\kappa\kappa' & \frac{g^2}{4}(\kappa^2+\kappa'^2+2\nu_R^2)\end{pmatrix}\begin{pmatrix}W_L^-\\ W_R^+\end{pmatrix}
\end{aligned}
\tag{3.38}
$$

where we use the equivalent

$$
W^2_{1L,R}+W^2_{2L,R}={W^+_{L,R}}^2+{W^-_{L,R}}^2=2W^+_{\mu L,R}W^{-,\mu}_{L,R}
\tag{3.39}
$$

The superscript 2 is for square and we have put the component indices $1,2,3$ into subscript for easy reading. Use similarity transformation $SMS^{-1}=M^{dia}$ to diagonalize the mixing matrix. Transformation matrix S belongs to $SO(2)$ for real mixing matrix. Gauge bosons transform as

$$
\begin{pmatrix}W_L^+\\ W_R^-\end{pmatrix}=\begin{pmatrix}\cos\xi_W & -\sin\xi_W\\ \sin\xi_W & \cos\xi_W\end{pmatrix}\begin{pmatrix}W_1^+\\ W_2^-\end{pmatrix}
\tag{3.40}
$$

$$
\begin{pmatrix}W_L^-\\ W_R^+\end{pmatrix}=\begin{pmatrix}\cos\xi_W & -\sin\xi_W\\ \sin\xi_W & \cos\xi_W\end{pmatrix}\begin{pmatrix}W_1^-\\ W_2^+\end{pmatrix}
\tag{3.41}
$$

where ξ_W defined as the mixing angle between two W matrix. Insert this back into the mass mixing term.

$$
\begin{aligned}
&\begin{pmatrix}\cos\xi_W & \sin\xi_W\\ -\sin\xi_W & \cos\xi_W\end{pmatrix}\begin{pmatrix}\frac{g^2}{4}(\kappa^2+\kappa'^2+2\nu_L^2) & -\frac{1}{2}g^2\kappa\kappa'\\ -\frac{1}{2}g^2\kappa\kappa' & \frac{g^2}{4}(\kappa^2+\kappa'^2+2\nu_R^2)\end{pmatrix}\begin{pmatrix}\cos\xi_W & -\sin\xi_W\\ \sin\xi_W & \cos\xi_W\end{pmatrix}\\
=&\begin{pmatrix}\mathbf{m}^{dia}_{11} & \mathbf{m}^{dia}_{12}\\ \mathbf{m}^{dia}_{21} & \mathbf{m}^{dia}_{22}\end{pmatrix}
\end{aligned}
\tag{3.42}
$$

where the entries are

$$
\begin{aligned}
\mathbf{m}^{dia}_{11}&=\frac{g^2}{4}(\kappa^2+\kappa'^2+2\nu_L^2)c_W^2+\frac{g^2}{4}(\kappa^2+\kappa'^2+2\nu_R^2)s_W^2-g^2\kappa\kappa' s_Wc_W\\
\mathbf{m}^{dia}_{22}&=\frac{g^2}{4}(\kappa^2+\kappa'^2+2\nu_L^2)s_W^2+\frac{g^2}{4}(\kappa^2+\kappa'^2+2\nu_R^2)c_W^2+g^2\kappa\kappa' s_Wc_W\\
\mathbf{m}^{dia}_{12}&=\mathbf{m}^{dia}_{21}=-\frac{g^2}{4}(\kappa^2+\kappa'^2+2\nu_L^2)s_Wc_W+\frac{g^2}{4}(\kappa^2+\kappa'^2+2\nu_R^2)s_Wc_W\\
&\qquad+\frac{1}{2}g^2\kappa\kappa'(s_W^2-c_W^2)
\end{aligned}
\tag{3.43}
$$

where s_W, c_W are abbreviations of $\sin\xi_W, \cos\xi_W$ respectively. The off-diagonal parts are expected to be zero $\mathbf{m}_{12}^{dia} = \mathbf{m}_{21}^{dia} = 0$. Dividing $\cos^2\xi_W$ on both sides and doing some algebra, we have

$$\tan 2\xi_W = \frac{\kappa\kappa'}{\nu_R^2 - \nu_L^2} \tag{3.44}$$

This is the diagonalization condition of $W^\pm$ mass mixing matrix. It is not hard to see that W_1^+, W_2^+ have the same mass as W_1^-, W_2^- respectively. Write down the masses expression with $\tan 2\xi_W$

$$\begin{aligned} M_{W_1}^2 &= \frac{g^2}{4}(\kappa^2 + \kappa'^2) + \frac{g^2}{2}\cos^2\xi_W(\nu_L^2 + \nu_R^2\tan^2\xi_W - 2\kappa\kappa'\tan\xi_W) \\ M_{W_2}^2 &= \frac{g^2}{4}(\kappa^2 + \kappa'^2) + \frac{g^2}{2}\cos^2\xi_W(\nu_L^2\tan^2\xi_W + \nu_R^2 + 2\kappa\kappa'\tan\xi_W) \end{aligned} \tag{3.45}$$

Next, let us decouple the mixing of the rest neutral fields in Equation (3.37).

$$\begin{aligned} &\frac{g^2}{4}(\kappa^2 + \kappa'^2 + 4\nu_L^2)W_{3L}^2 + \frac{g^2}{4}(\kappa^2 + \kappa'^2 + 4\nu_R^2)W_{3R}^2 - \frac{g^2}{2}(\kappa^2 + \kappa'^2)W_{3L}W_{3R} \\ &- 2gg'(\nu_L^2 W_{3L,\mu}B_\mu + \nu_R^2 W_{3R,\mu}B^\mu) + g'^2(\nu_L^2 + \nu_R^2)B_\mu B^\mu \\ &= \begin{pmatrix} W_{3L} \\ W_{3R} \\ B \end{pmatrix}^T \begin{pmatrix} \frac{g^2}{4}(\kappa^2 + \kappa'^2 + 4\nu_L^2) & \frac{g^2}{4}(\kappa^2 + \kappa'^2) & -gg'\nu_L^2 \\ \frac{g^2}{4}(\kappa^2 + \kappa'^2) & \frac{g^2}{4}(\kappa^2 + \kappa'^2 + 4\nu_R^2) & -gg'\nu_R^2 \\ -gg'\nu_L^2 & -gg'\nu_R^2 & g'^2(\nu_L^2 + \nu_R^2) \end{pmatrix} \begin{pmatrix} W_{3L} \\ W_{3R} \\ B \end{pmatrix} \end{aligned} \tag{3.46}$$

The transformation of the field states has the same structure as the three generations mixing. Using the standard parametrization without the phase shift, the transformation is

$$\begin{pmatrix} Z_1 \\ Z_2 \\ A \end{pmatrix} = \begin{pmatrix} c_{13}c_{12} & c_{13}s_{12} & s_{13} \\ -s_{23}s_{13}c_{12} - c_{23}s_{12} & -s_{23}s_{13}s_{12} + c_{12}c_{23} & s_{23}c_{13} \\ -c_{23}s_{13}c_{12} + s_{23}s_{12} & -c_{23}s_{13}s_{12} - s_{23}c_{12} & c_{23}c_{13} \end{pmatrix} \begin{pmatrix} W_{3L} \\ W_{3R} \\ B \end{pmatrix} \tag{3.47}$$

where the shorthand is the same as Equation (2.83). It is not easy to diagonalize Equation (3.46) directly. Besides, we can make some approximations to the VEVs to simplify the mass matrix. The masses of charged W bosons as well as the neutral bosons masses in a glance of (3.46) are symmetric in exchange of left and right indices, which means they have parity conservation. One of the methods to generate the chirality asymmetry is to suppose that the VEVs are in different hierarchy of the mass scales. From the mLRSM to the SM, the right-handed gauge fields hide away and only the left-handed fields remain. In this circumstances, the right-handed VEV are assumed to be much larger than the left-handed VEV, i.e. $\nu_R \gg \nu_L$, and the mixing between W_L and W_R can be ignored. This approximation will be reconsidered in the discussion of neutrino masses. Now let us admit the approximations: (1) $\theta_{12} = 0$ in Equation (3.47) when we ignore the W_L, W_R mixing; (2) $\nu_R \gg \nu_L, \nu_R \gg \kappa \approx \kappa'$.

The transformation matrix (3.47) then goes to

$$\begin{pmatrix} Z_1 \\ Z_2 \\ A \end{pmatrix} = \begin{pmatrix} c_{13} & 0 & s_{13} \\ -s_{23}s_{13} & c_{23} & s_{23}c_{13} \\ -c_{23}s_{13} & -s_{23} & c_{23}c_{13} \end{pmatrix} \begin{pmatrix} W_{3L} \\ W_{3R} \\ B \end{pmatrix} \tag{3.48}$$

and solve the characteristic function to obtain the eigenvalues

$$\begin{vmatrix} g^2\nu_L^2 - \lambda & 0 & -g'\nu_L^2 \\ 0 & g^2\nu_R^2 - \lambda & -gg'\nu_R^2 \\ -gg'\nu_L^2 & -gg'\nu_R^2 & g'^2\nu_R^2 - \lambda \end{vmatrix} = 0 \tag{3.49}$$

and the eigenvalues which are the masses of Z_1, Z_2, A are

$$m_A = 0 \tag{3.50}$$

$$m_{Z_1} = \frac{1}{2}(g^2\nu_L^2 + g'^2\nu_L^2 + g^2\nu_R^2 + g'^2\nu_R^2) - \frac{1}{2}\sqrt{\Delta_d} \tag{3.51}$$

$$m_{Z_2} = \frac{1}{2}(g^2\nu_L^2 + g'^2\nu_L^2 + g^2\nu_R^2 + g'^2\nu_R^2) + \frac{1}{2}\sqrt{\Delta_d} \tag{3.52}$$

where the discriminant Δ_d is

$$\Delta_d = (g^2\nu_L^2 + gp^2\nu_L^2 + g^2\nu_R^2 + g'^2\nu_R^2)^2 - 4(g^4\nu_L^2\nu_R^2 + 2g^2g'^2\nu_L^2\nu_R^2) \tag{3.53}$$

then use the approximation $\nu_L \ll \nu_R$, it comes to

$$m_{Z_1} \approx \frac{1}{2}(g^2{\nu_L}^2 + g'^2\nu_L^2) \tag{3.54}$$

$$m_{Z_2} \approx g^2\nu_R^2 + g'^2\nu_R^2 \tag{3.55}$$

where in our approximations, the mass of the light Z boson depends on the left-handed Higgs scalar VEV ν_L and the heavy one is related to ν_R.

3.4 Quark Sector

The only contribution for quark mass terms is,

$$\begin{aligned} &-\sum_{i,j}^{3}(\bar{Q}_{Li}\Gamma_{ij}^{Q}\phi Q_{Rj} + \bar{Q}_{Li}\tilde{\Gamma}_{ij}^{Q}\tilde{\phi}Q_{Rj} + \bar{Q}_{Ri}\phi^\dagger\Gamma_{ij}^{Q\dagger}Q_{Lj} + \bar{Q}_{Ri}\tilde{\phi}\tilde{\Gamma}_{ij}^{Q\dagger}Q_{Lj}) \\ &= -\kappa\begin{pmatrix}\overline{u_L} & \overline{c_L} & \overline{t_L}\end{pmatrix}\mathbf{\Gamma}^u\begin{pmatrix}u_R & c_R & t_R\end{pmatrix}^T - \kappa'\begin{pmatrix}\overline{d_L} & \overline{s_L} & \overline{b_L}\end{pmatrix}\mathbf{\Gamma}^d\begin{pmatrix}d_R & s_R & b_R\end{pmatrix}^T \\ &\quad -\kappa'\begin{pmatrix}\overline{u_L} & \overline{c_L} & \overline{t_L}\end{pmatrix}\tilde{\mathbf{\Gamma}}^u\begin{pmatrix}u_R & c_R & t_R\end{pmatrix}^T - \kappa\begin{pmatrix}\overline{d_L} & \overline{s_L} & \overline{b_L}\end{pmatrix}\tilde{\mathbf{\Gamma}}^d\begin{pmatrix}d_R & s_R & b_R\end{pmatrix}^T \\ &\quad + h.c. \end{aligned} \tag{3.56}$$

where $\mathbf{\Gamma}, \tilde{\mathbf{\Gamma}}$ are 3 by 3 matrices (Yukawa matrices), and the superscript u, d are for up types and down types. Define the transformation to the Lagrangian from flavor states to masses eigenstates. The mass eigenstates of the quarks are $q^u_{L,R} = (u,c,t)^T_{L,R}$ and $q^d_{L,R} = (d,s,b)^T_{L,R}$.

$$\begin{aligned} q^u_{L,R} &= {U^u_{L,R}}^\dagger Q^u_{L,R} & \qquad q^d_{L,R} &= {U^d_{L,R}}^\dagger Q^d_{L,R} \\ \bar{q}^u_{L,R} &= \bar{Q}^u_{L,R} U^u_{L,R} & \qquad \bar{q}^d_{L,R} &= \bar{Q}^d_{L,R} U^d_{L,R} \end{aligned} \tag{3.57}$$

The matrices ${U^u_{L,R}}^\dagger U^u_{L,R} = {U^d_{L,R}}^\dagger U^d_{L,R} = \mathbb{1}$ are unitary matrices for diagonalization transformation. These are part of the CKM matrix $V^{CKM} \equiv U^u_L U^{d\dagger}_L$ (Cabibbo–Kobayashi–Maskawa matrix [68, 69]) which is defined from the quark current, i.e.

$$\mathscr{L}^{cc}_q = -\bar{q}^u_L \gamma^\mu V^{CKM} q^d_L W^+_\mu + h.c. \tag{3.58}$$

This allows us to make an assumption in the left-right symmetric scenario: $V^{CKM} \equiv U^u_L U^{d\dagger}_L = U^u_R U^{d\dagger}_R$, the right-handed transformation matrix is identical to the left-handed one. The diagnolization procedure is very similar with what we have done in the SM. But we have both right-handed and left-handed doublets now.

$$\begin{aligned} &-\kappa \bar{q}^u_L U^{u\dagger}_L \mathbf{\Gamma}^u U^u_R q^u_R - \kappa' \bar{q}^d_L U^{d\dagger}_L \mathbf{\Gamma}^d U^d_R q^d_R - \kappa' \bar{q}^u_L U^{u\dagger}_L \tilde{\mathbf{\Gamma}}^u U^u_R q^u_R - \kappa \bar{q}^d_L U^{d\dagger}_L \tilde{\mathbf{\Gamma}}^d U^d_R q^d_R + h.c. \\ &= -\,\bar{q}^u_L(\kappa U^{u\dagger}_L \mathbf{\Gamma}^u U^u_R + \kappa' U^{u\dagger}_L \tilde{\mathbf{\Gamma}}^u U^u_R) q^u_R - \bar{q}^d_L(\kappa U^{d\dagger}_L \tilde{\mathbf{\Gamma}}^d U^d_R + \kappa' U^{d\dagger}_L \mathbf{\Gamma}^d U^d_R) q^d_R + h.c \end{aligned} \tag{3.59}$$

And the diagonalization of the $\mathbf{\Gamma}$ matrices are

$$U^{u,d\dagger}_L \mathbf{\Gamma}^{u,d} U^{u,d}_R \equiv M^{diag}_{u,d} \qquad\qquad U^{u,d\dagger}_L \tilde{\mathbf{\Gamma}}^{u,d} U^{u,d}_R \equiv \tilde{M}^{diag}_{u,d} \tag{3.60}$$

where the parameters of the transformation follow the discussion of the standard parametrization. The masses of up quarks and down quarks are proportional to the VEVs,

$$\begin{aligned} m_{u,c,t} &= \kappa M^{diag}_{u,c,t} + \kappa' \tilde{M}^{diag}_{u,c,t} \\ m_{d,s,b} &= \kappa' M^{diag}_{d,s,b} + \kappa \tilde{M}^{diag}_{d,s,b} \end{aligned} \tag{3.61}$$

Thus, for the not-exact $SU(2)$ symmetry of u-quark and d-quark, the mass scale of quarks $\kappa + \kappa'$ is much smaller than the mass scale of right-handed bosons ν_R under the assumption in the last section. The very high masses scales of top quark and bottom quark can be provided by the Yukawa couplings.

3.5 Lepton Sector

There are two types of mass terms that contribute to leptons. One is the Dirac masses that couple with the bidoublet Higgs field, and the other one is the Majorana masses couple with left- and right-handed Higgs triplets. The mass scale of neutrinos of any BSMs is a crucial property to combine experiments and theories. In the SM symmetry, the pure Dirac neutrinos are massless due to the lepton number conservation and global gauge invariance. The existence of massive neutrinos is consistent with the lepton number violation. The lower bound of the massive neutrino mass scale shows the observation possibility of these neutrinos. For other reasons, the heavy neutrinos could be a hint for why light neutrino masses are much smaller compared to the charged leptons. We have seen in the previous section that the smallness of the light neutrino masses under the seesaw mechanism is generated due to the large mass of the heavy neutrino. Now let us derive the expression of the masses of charged leptons and neutrinos in the mLRSM.

The Dirac neutrino masses

The Dirac neutrino masses are available for $\nu_L\nu_R + \nu_R\nu_L$ term. The derivation procedure is the same as the one for quarks in section 3.4. But the transformation we introduce here should have different values in the entries. They form the PMNS matrix in the lepton charged current, as we have done in the SM case, see Equation (2.74). The mass terms using the mass basis are

$$-\overline{\nu}_{iL}(\kappa M_D^\nu + \kappa' \tilde{M}_D^\nu)\nu_{iR} - \overline{l}_L(\kappa M_D^l + \kappa' \tilde{M}_D^l)l_R + h.c. \tag{3.62}$$

where the diagonalized mass matrices for the neutrino mass are

$$M_D^\nu \equiv U_L^{\nu\dagger}\Gamma^\nu U_R^\nu \qquad \tilde{M}_D^\nu \equiv U_L^{\nu\dagger}\tilde{\Gamma}^\nu U_R^\nu \tag{3.63}$$

$$M_D^l \equiv U_L^{l\,\dagger}\Gamma^l U_R^l \qquad \tilde{M}_D^l \equiv U_L^{l\,\dagger}\tilde{\Gamma}^l U_R^l \tag{3.64}$$

The transformation matrices have the same structure as those of the SM. The PMNS matrix from the lepton-$W_{LR}^\pm$ current is: $U^{PMNS} = U_L^{l\dagger}U_L^\nu = U_R^{l\dagger}U_R^\nu$ (the left- and right-handed transformation matrices are equivalent) [30]. There is no difference for how the three flavor generations ν_e, ν_μ, ν_τ mixing from different models, unless other type of neutrinos are taken into account. We have the Dirac neutrino masses and charge-lepton masses [70]

$$m_{\nu_i} = \kappa M_{D,i}^\nu + \kappa' \tilde{M}_{D,i}^\nu \tag{3.65}$$

$$m_{l_i} = \kappa M_{D,i}^l + \kappa' \tilde{M}_{D,i}^l \tag{3.66}$$

where the subscript i runs over $1, 2, 3$ of the mass eigenstates and the corresponding entries. The pure Dirac neutrino mass depends on the VEVs of the Higgs bidoublet and the Yukawa couplings. The masses of charged leptons differ from the neutrinos' only in Yukawa couplings. One can choose a condition where charged leptons and Dirac neutrinos have a similar mass scale.

The Majorana neutrino masses

From Equation (2.25) to (2.28), we introduce the charge conjugate operator and the Majorana field. The charge conjugate operator is representation dependent, and in this thesis we are using Dirac spinors. Remember, the gamma matrices and the charge conjugate matrix only act on the neutrino wave function under the Lorentz transformation (for the wave function $\phi(x)$ for example, there are two spinors solution of Dirac equation and for each there are two components, so 4 in total). We do not need to consider it explicitly in the Higgs mechanism, where the $SU(2)_{L,R}$ doublets couple the $SU(2)$ triplet $\Delta_{L,R}$. The Majorana terms with the Hermitian conjugates of Equation (3.2) are

$$\begin{aligned}\mathscr{L}_Y^M = &- \sum_{Leptons} \left\{G_{L,ij}\overline{L_{Li}^c}i\sigma_2\Delta_L L_{Lj} + G_{R,ij}\overline{L_{Ri}^c}i\sigma_2\Delta_R L_{Rj}\right\} \\ &- \sum_{leptons} \left\{G_{L,ij}\overline{L_{L,i}}i\sigma_2\Delta_L(L_{L,j})^c + G_{R,ij}\overline{L_{R,i}}i\sigma_2\Delta_R(L_{R,j})^c\right\}\end{aligned} \tag{3.67}$$

where we directly use $i\sigma_2\Delta_{L,R}$ instead of $\vec{\sigma}\cdot\vec{\Delta_{LR}}$ since only the σ_2 component remains when entering the VEVs. σ_2 is the Pauli matrix with imaginary entries. Here we take: $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We also have $i\sigma_2\delta_L = \begin{pmatrix} \delta_L^0 & -\frac{1}{\sqrt{2}}\delta_L^+ \\ -\frac{1}{\sqrt{2}}\delta_L^+ & -\delta_L^{++} \end{pmatrix}$ and the right-handed counterpart. The explicit form of (3.67) contains a 6×6 matrix

$$\begin{aligned}\mathscr{L}_Y^M = &- \begin{pmatrix}\overline{\nu_{eL}^c} & \overline{\nu_{\mu L}^c} & \overline{\nu_{\tau L}^c} & \overline{e_L^c} & \overline{\mu_L^c} & \overline{\tau_L^c}\end{pmatrix} \mathbf{G_{L6\times6}} \begin{pmatrix}\nu_{eL} & \nu_{\mu L} & \nu_{\tau L} & e_L & \mu_L & \tau_L\end{pmatrix}^T \\ &- \begin{pmatrix}\overline{\nu_{eR}^c} & \overline{\nu_{\mu R}^c} & \overline{\nu_{\tau R}^c} & \overline{e_R^c} & \overline{\mu_R^c} & \overline{\tau_R^c}\end{pmatrix} \mathbf{G_{R6\times6}} \begin{pmatrix}\nu_{eR} & \nu_{\mu R} & \nu_{\tau R} & e_R & \mu_R & \tau_R\end{pmatrix}^T \\ &+ h.c.\end{aligned} \tag{3.68}$$

where the 6×6 matrices $\mathbf{G_{L6\times6}}, \mathbf{G_{R6\times6}}$ are,

$$\mathbf{G_{L6\times6}} = \begin{pmatrix} \mathbf{G_{11}}_L\delta_L^0\mathbb{1}_{3\times3} & \mathbf{G_{12}}_L(-\frac{1}{\sqrt{2}}\delta_L^+\mathbb{1}_{3\times3}) \\ \mathbf{G_{21}}_L(-\frac{1}{\sqrt{2}}\delta_L^+\mathbb{1}_{3\times3}) & \mathbf{G_{22}}_L(-\delta_L^{++}\mathbb{1}_{3\times3}) \end{pmatrix} \tag{3.69}$$

$$\mathbf{G_{R6\times6}} = \begin{pmatrix} \mathbf{G_{11}}_R\delta_R^0\mathbb{1}_{3\times3} & \mathbf{G_{12}}_R(-\frac{1}{\sqrt{2}}\delta_R^+\mathbb{1}_{3\times3}) \\ \mathbf{G_{21}}_R(-\frac{1}{\sqrt{2}}\delta_R^+\mathbb{1}_{3\times3}) & \mathbf{G_{22}}_R(-\delta_R^{++}\mathbb{1}_{3\times3}) \end{pmatrix} \tag{3.70}$$

$\mathbf{G_{11}}_L, \mathbf{G_{12}}_L, \mathbf{G_{21}}_L, \mathbf{G_{22}}_L$ is the Yukawa coupling matrices, and each of them is a 3×3 matrix. The first block in the block matrix produces the mass matrix for neutrinos. As we can see, the charged leptons do not couple with the first block, which means that they do not contain the Majorana mass. The fourth block matrix $\mathbf{G_{22}}_L$ generates the interaction of the charged leptons and the charged Higgs scalars, where we can calculate the coupling of the lepton-Higgs vertex. The vertex depends on the Yukawa matrix,

$$\mathscr{L}_Y^{l-h} = \sum_{\alpha,\beta} G_{22L,\alpha\beta}\delta_L^{++}\overline{l_{L,\alpha}^c}l_{L,\beta} + \sum_{\alpha,\beta} G_{22R,\alpha\beta}\delta_R^{++}\overline{l_{R,\alpha}^c}l_{R,\beta} + h.c. \tag{3.71}$$

where $\alpha, \beta = e, \mu, \tau$. In the SM, we have an assumption that the Dirac neutrino mass is at the scale level of charged leptons. Here we can reach this by assuming that all the Yukawa matrices are equivalent, i.e. $\mathbf{G_{11}}_L = \mathbf{G_{12}}_L = \mathbf{G_{21}}_L = \mathbf{G_{22}}_L \equiv \mathbf{G}_L$. In order to determine the Yukawa matrix, we require the mass terms with the VEVS inserting into the Lagrangian. Let us explicitly calculate this:

$$\begin{aligned}\mathscr{L}_Y^M =& -\frac{1}{2}\nu_L \begin{pmatrix}\overline{\nu_{eL}^c} & \overline{\nu_{\mu L}^c} & \overline{\nu_{\tau L}^c}\end{pmatrix} \mathbf{G}_L \begin{pmatrix}\nu_{eL} & \nu_{\mu L} & \nu_{\tau L}\end{pmatrix}^T \\ & -\frac{1}{2}\nu_R \begin{pmatrix}\overline{\nu_{eR}^c} & \overline{\nu_{\mu R}^c} & \overline{\nu_{\tau R}^c}\end{pmatrix} \mathbf{G}_R \begin{pmatrix}\nu_{eR} & \nu_{\mu R} & \nu_{\tau R}\end{pmatrix}^T + h.c. \\ =& -\frac{1}{2}\nu_L M_L^M \sum_i^3 (\overline{\nu_{iL}^c}\nu_{iL} + \overline{\nu_{iL}}(\nu_{iL})^c) - \frac{1}{2}\nu_R M_R^M \sum_i^3 (\overline{\nu_{iR}^c}\nu_{iR} + \overline{\nu_{iR}}(\nu_{iR})^c)\end{aligned} \tag{3.72}$$

where M^M is the diagonalized matrix from $\mathbf{G}$, and since the flavor mixing matrix $\mathbf{G}$ is supposed to be real, the diagonalizations of $\mathbf{G}$ and $\mathbf{G}^\dagger$ give the same result: $M_L^M = {U^l}^\dagger_L \mathbf{G}_L U_L^l$. In the model without parity asymmetry, the Yukawa matrices G_L and G_R can be chosen to be the same. But we will still have the subscripts in the later derivations for clearer reading. The mass couplings of the left- and right-handed neutrinos in the field theory perception are $\nu_L M_L^M, \nu_R M_R^M$ respectively. The Yukawa matrix can then be written as

$$\mathbf{G}_{L,R} = U M_{L,R}^{diag} U^\dagger \tag{3.73}$$

where U is the neutrino mixing matrix. Continue from Equation (3.72) and define a combination of the neutrino fields and its conjugate,

$$\begin{aligned} N_L &= \nu_L + \nu_L^c & \qquad N_R &= \nu_R + \nu_R^c \\ \overline{N_L^c} &= \overline{\nu_L^c} + \overline{\nu_L} & \qquad \overline{N_R^c} &= \overline{\nu_R^c} + \overline{\nu_R} \end{aligned} \tag{3.74}$$

The Majorana mass terms now can be written as

$$\mathscr{L}_Y^M = -\frac{1}{2}\nu_L M_L^M \sum_i^3 \overline{N_{iL}^c} N_{iL} - \frac{1}{2}\nu_R M_R^M \sum_i^3 \overline{N_{iR}^c} N_{iR} \tag{3.75}$$

Therefore, the masses scales of left(right)-handed neutrinos are proportional to VEVs $\nu_L(\nu_R)$ and the Yukawa couplings. Let us calculate a bit more, to check if the fields we define here are really Majorana fields, i.e. $CN_LC = C\overline{N_L}^T \stackrel{!}{=} N_L$.

By doing this, we take the charge conjugate to N_L,

$$\begin{aligned} CN_LC &= C\overline{\nu_L}^T = C\overline{\nu_L}^T + C\overline{\nu_L^c}^T = \overline{\overline{\nu_L}^T}^T + C(-\nu_L^T C^{-1})^T \\ &= \overline{\overline{\nu_L}^T}^T + C(-(C^{-1})^T\nu_L) = \nu_L^c + \nu_L = N_L \end{aligned} \tag{3.76}$$

in the last line we use the fact that $(C^{-1})^T = -C^{-1}$. With the same reason in the SM, we now put the Dirac neutrino terms and the Majorana neutrino terms together. For simplicity, the summation symbol has been removed. (I have to apologize for the inconvenience of notation; $\nu_{L,R}$ are VEVs of Higgs triplet and those with subscript i are Dirac neutrino fields.) We can write $\nu_R \equiv \nu_L^c$ from Equation (2.28). The Lagrangian would be

$$\begin{aligned} \mathscr{L}_Y^\nu =& -\overline{\nu_{iL}}(\kappa M_D^\nu + \kappa'\tilde{M}_D^\nu)\nu_{iR} - \overline{\nu_{iR}}(\kappa M_D^\nu + \kappa'\tilde{M}_D^\nu)\nu_{iL} \\ & - \frac{1}{2}\nu_L M_L^M \overline{N_{iL}^c} N_{iL} - \frac{1}{2}\nu_R M_R^M \overline{N_{iR}^c} N_{iR} \\ =& -\frac{1}{2}\begin{pmatrix}\overline{\nu_{iL}^c} & \overline{\nu_{iR}}\end{pmatrix}\begin{pmatrix}\nu_L M_L^M & (\kappa M_D + \kappa'\tilde{M}_D) \\ (\kappa M_D + \kappa'\tilde{M}_D) & \nu_R M_R^M\end{pmatrix}\begin{pmatrix}\nu_{iL} \\ \nu_{iR}^c\end{pmatrix} \\ & -\frac{1}{2}\begin{pmatrix}\overline{\nu_{iL}} & \overline{\nu_{iR}^c}\end{pmatrix}\begin{pmatrix}\nu_L M_L^M & (\kappa M_D + \kappa'\tilde{M}_D) \\ (\kappa M_D + \kappa'\tilde{M}_D) & \nu_R M_R^M\end{pmatrix}\begin{pmatrix}\nu_{iL}^c \\ \nu_{iR}\end{pmatrix} \end{aligned} \tag{3.77}$$

where we ignore the superscript ν of the Dirac mass matrix. The mass matrix is a 6×6 matrix mixed by the Dirac and Majorana neutrinos. This mixing mass matrix has the same structure and symmetry as the one in the SM, i.e. "$SU(2)_L \times U(1)_Y$ with right-handed singlet fixing". In the mLRSM, there are right-handed counterpart contributions for Dirac es, which come from the "Higgs bidoublet- lepton" interaction.

The Seesaw Mechanism

Equation (3.77) can be diagonalized to light and heavy neutrinos using the seesaw mechanism. This is called the type II seesaw mechanism, where the Higgs triplet with heavy VEVs has been taken into account. First solve the eigenfunction of the mass matrix. The exact eigenvalues are

$$\begin{aligned} m_{\nu_1} =& \frac{1}{2}(\nu_L M_L^M + \nu_R M_R^M) - \frac{1}{2}\sqrt{\mathbf{\Delta}'} \\ m_{\nu_2} =& \frac{1}{2}(\nu_L M_L^M + \nu_R M_R^M) + \frac{1}{2}\sqrt{\mathbf{\Delta}'} \end{aligned} \tag{3.78}$$

where $\mathbf{\Delta}'$ is the discriminant,

$$\mathbf{\Delta}' = (\nu_L M_L^M + \nu_R M_R^M)^2 - 4\nu_L\nu_R M_L^M M_R^M(\kappa M_D + \kappa'\tilde{M}_D)(\kappa M_D + \kappa'\tilde{M}_D) \tag{3.79}$$

Let us determine the transformation matrix, and write the general diagonalization of any real 2 by 2 matrix.

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \nu_R M_R^M \begin{pmatrix} b & a \\ a & c \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{3.80}$$

where the shorthand a, b, c is

$$a = \kappa M_D + \kappa' \tilde{M}_D \qquad b = \nu_L M_L^M \qquad c = \nu_R M_R^M \tag{3.81}$$

We set the off-diagonal part to zero in order to find the mixing angle θ relation.

$$\tan 2\theta = \frac{2a}{c-b} \qquad \cos 2\theta = \frac{c-b}{\sqrt{4a^2 + (c-b)^2}} \tag{3.82}$$

where we consider the same approximations of the VEVs $\nu_L \ll \nu_R, \kappa + \kappa' \ll \nu_R$, and this leads to $c \gg a, c \gg b$. With these conditions, the diagonalization matrix could be written as

$$\begin{pmatrix} 1 & \frac{\sqrt{2}a}{c-b} \\ -\frac{\sqrt{2}a}{c-b} & 1 \end{pmatrix} \tag{3.83}$$

then the mass diagonalized matrix is

$$\begin{pmatrix} b - \frac{2\sqrt{2}a^2}{c-b} & 0 \\ 0 & c + \frac{2\sqrt{2}a^2}{c-b} \end{pmatrix} \tag{3.84}$$

Put back the exact expressions of the shorthand. The masses of neutrinos through the seesaw mechanism are

$$\begin{aligned} m_{\nu_i} &\approx \nu_L M_L^M - 2\sqrt{2}(\kappa M_D + \kappa' \tilde{M}_D)^T (\nu_R M_R^M)^{-1} (\kappa M_D + \kappa' \tilde{M}_D) \\ m_{N_i} &\approx \nu_R M_R^M + 2\sqrt{2}(\kappa M_D + \kappa' \tilde{M}_D)^T (\nu_R M_R^M)^{-1} (\kappa M_D + \kappa' \tilde{M}_D) \end{aligned} \tag{3.85}$$

or make a further estimation ($\nu_L \approx 0$)

$$m_{\nu_i} \approx \frac{(\kappa M_D + \kappa' \tilde{M}_D)^2}{\nu_R M_R^M} \qquad m_{N_i} \approx \nu_R M_R^M \tag{3.86}$$

where i is for three generations. M is the short for M_i that gives the entries of the Yukawa matrix couplings. The seesaw mechanism we have done here is based on the same mathematics as the SM's process. For any mixing of two components of fields, we could do the same procedure. For some other models which introduce additional fields coupled with neutrinos, the mass matrix will be enlarged to a 3 by 3 matrix or even to higher dimensions. We will not discuss them here since it is not the major topic of this thesis, but one can still take a quick glance at section (3.6).

The Equation (3.86) implies that the light neutrino-heavy neutrino ratio is approximately

$$\frac{m_{\nu_i}}{m_{N_i}} = \frac{(\kappa M_D + \kappa' \tilde{M}_D)^2}{(\nu_R M_R^M)^2} \tag{3.87}$$

when compare to Equations (3.66) and (3.38) we come to the conclusion that the ratio in mLRSM under the approximation $\nu_R \gg \nu_L \approx 0, \nu_R \gg \kappa/\kappa'$ is rewritten in this way:

$$\frac{m_{\nu_i}}{m_{N_i}} \approx \frac{m_l^2}{m_{W_R}^2} \tag{3.88}$$

where the mass scale of the right-handed gauge boson is proportional to ν_R. Consider the electron neutrino, the ratio becomes

$$\frac{m_{\nu_e}}{m_{N_e}} \approx \frac{m_e^2}{m_{W_R}^2} \tag{3.89}$$

It is not forbidden to choose a heavy neutrino around $m_{N_i} \approx 230$GeV [60, 64]. If we want the light neutrino mass smaller than 1eV, we need the right-handed W boson mass scale to be $m_{W_R} \approx 240$GeV from Equation (3.89). Therefore, with the data from the SM experiments for the mass of the left-handed W boson [71], we have $m_{W_R} \gtrsim 3m_{W_L}$. The choice of the heavy neutrino mass is then consist to $m_{N_i} \approx m_{W_R} \approx \nu_R$ [60, 72].

We can also choose different heavy boson scenarios. In the above seesaw mechanism, usually we choose ν_R to be very large, and this leads to a very large right-handed gauge boson comparing to the left one $m_{W_R} \gg m_{W_L}$, so that we can obtain a naturally vanished neutrino mass [60]. In the assumption where it is only $\nu_R \gg \kappa + \kappa'$, and with the W gauge boson mass (3.45) $m_{W_L} \approx \kappa + \kappa'$, we have the neutrino mass ratio $m_\nu \approx \frac{m_{W_L}^2}{m_{W_R}^2}$. Therefore, when the heavy neutrino mass is supposed to be at the limit of 1GeV, in order to have the light neutrino mass $\lesssim$ 1eV, it requires a very heavy right gauge boson $m_{W_R} \approx 10^4 m_{W_L}$ to $m_{W_R} \approx 10^5 m_{W_L}$.

3.6 Other Left-Right Symmetric Scenarios

Here we introduce some other LRSMs which have parity restoration at higher symmetry in common. Since the Majorana neutrino and the neutrinoless double decay are not directly detected, one can also imagine a model without Majorana particles. For instance, (1) in the paper [73], the authors construct an alternative global $B-L$ number conserved model with two Higgs doublets $\chi_{L(R)} \equiv [1, 1(2), 2(1), 1]$ instead of one Higgs bidoublet. There is also a pure scalar field introduced to enlarge the symmetry by another $U(1)_\chi$ where they define an additional quantum number ζ to compensate in $X = B - L + \zeta$. The Dirac es are generated from a two-loop radioactive diagram and a Dirac seesaw mechanism. (2) In [74], two leptoquarks and their duals in left-right associated with a new global quantum number have been introduced. The decaying process of the leptoquarks can explain the baryon number asymmetry even there is no massive Majorana neutrinos. The neutrino mass is generated through a Dirac type seesaw mechanism. However, these models under the minimal left-right symmetry do not approach the hierarchy problem of the standard electroweak and beyond [75]. In some other models, the parity violation is related to the spontaneous breaking of baryon and lepton numbers, so as to satisfy the experiment bounds on proton decay [76]. What's more, there are models with higher dimensions of symmetry groups, especially the enlarging of the generations such as $SU(4') \times SU(4)_L \times SU(4)_R$ [56]. Furthermore, there are symmetries in even higher dimensions, the so-called SUSY (supersymmetry) models, such as MSSM (minimal supersymmetric extension of the standard model) [77, 78].

4 Neutrinoless Double Beta Decay

4.1 Review of Double Beta Decay

Before we discuss double beta decay, it is necessary to start from the review of the single beta decay. The single beta decay is based on the weak interaction where a neutron transits to a proton state while the mother nucleus emits one electron and one electron antineutrino. The process can be written at the nuclear level as

$$(A, Z) \longrightarrow (A, Z+1) + e^- + \overline{\nu}_e \tag{4.1}$$

where A and Z are the nuclear mass number and the nuclear charge number (or proton number), respectively. If the mass of the daughter nucleus $M(A, Z+1)$ is less than the mother nucleus $M(A, Z)$, the decay process is permitted kinetically. The mass excess difference between them includes 1) the nuclear binding energy difference in various charge number Z with same mass number A; 2) the odd or even character of neutrons and protons in composition; and 3) the nuclear shell model [79].

$$M(A, Z) - M_A = \frac{1}{2} B_A (Z - Z_A)^2 + P_A - S \tag{4.2}$$

where M_A is the mass excess of the stable nucleus when $Z = Z_A$. The difference in binding energies with respect to Z is a parabolic form, and it is called the semiempirical equation [80]. S is the shell model correction term. P_A is specified as

$$P_A \propto \begin{cases} -1 & \text{even-even} \\ 0 & \text{even-odd} \\ +1 & \text{odd-odd} \end{cases} \tag{4.3}$$

where "even-odd" refers to even proton number and odd neutron number $N = A - Z$. In beta decays, the mass number A of the nucleus is identical before and after the decay. This will lead to two parabolic functions for the even mass number (even-even and odd-odd). In the odd mass number case, there is only one parabolic function. The transitions from the neighbor nuclei always have one proton number in difference and thereby the single beta decay is the highest possible among other format of beta decays. However, when the mother nuclei have even A number, two neighbors with one proton number different come from two parabolas (odd-odd and even-even), see Figure 4.1. This time, the single beta decay may not be the highest possible process. When single beta decay is kinetically forbidden, double beta decay

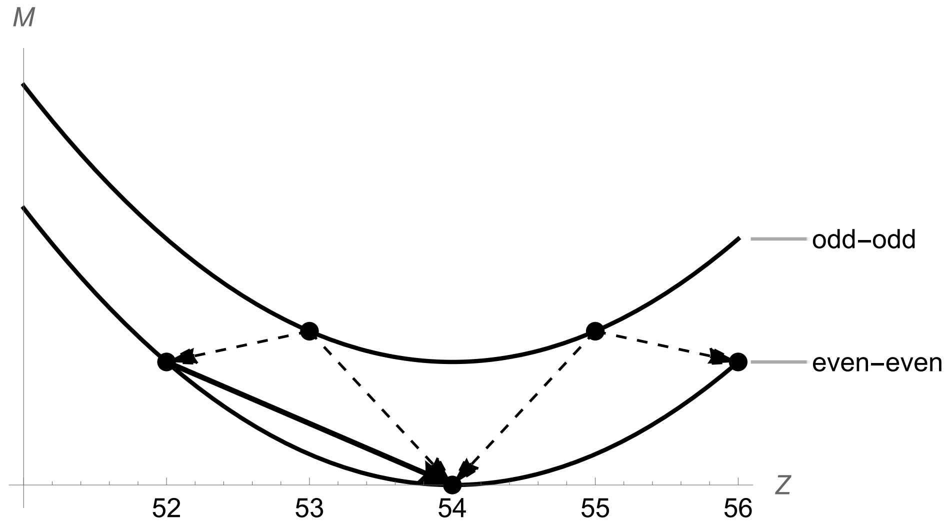


Figure 4.1: An example of even mass number beta decay where $A = 130$. The decay of the mother nucleus $(A, Z) = (130, 53)$ can decay into two daughter nuclei $(130, 52)$ and $(130, 54)$. The mass energy of $Z = 53$ one next to the minimum is higher than the second neighbor of the minimum $Z = 52$, which means the single beta decay from $Z = 52$ to the $Z = 53$ cannot happen spontaneously, instead the decay from nucleus $(130, 52)$ to the minimum state provides possibility, or in another word, single beta decay is prohibited.

has a chance to be dominant. The beta decay processes which change two proton number per time such as $(130, 52) \longrightarrow (130, 54)$ are called double beta decay. There are several types of double decay allowed by the SM mechanism [81].

$$\begin{aligned}
\beta^-\beta^- &: (A, Z) \longrightarrow (A, Z+2) + 2e^- + 2\overline{\nu}_e \\
\beta^+\beta^+ &: (A, Z) \longrightarrow (A, Z-2) + 2\overline{e}^+ + 2\nu_e \\
\mathrm{ECEC} &: 2e^- + (A, Z) \longrightarrow (A, Z-2) + 2\nu_e \\
\mathrm{EC}\beta^+ &: e^- + (A, Z) \longrightarrow (A, Z-2) + e^+ + 2\nu_e
\end{aligned} \tag{4.4}$$

The first process acts like two single beta decay adding together. The diagram is in Figure 4.2. EC stands for electron capture process. Although the double beta decay process is naturally allowed, the half-lives of the processes are extremely long. The observations in these decades show that the half-lives range from 10^{19} to 10^{24} years. Here are some example isotopes, see Table 4.1 [55].

$T_{\frac{1}{2}}$ is the half-life of the process. Half-lives are defined as the time period from the original substance decaying to the half amount of the substance. Assume that we have $N(t)$ unstable nuclei at some time t. The number of decay nuclei in an infinitesimal time period dt is proportional to the total number of unstable nuclei

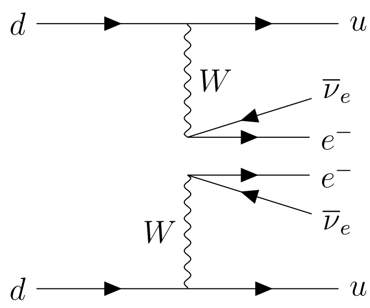


Figure 4.2: $\beta^-\beta^-$ decay

in t [84].

$$\frac{dN(t)}{dt} = -\Gamma N(t) \tag{4.5}$$

where Γ is the transition probability or decay constant. This constant depends on the specific nuclear type as well as on the initial and final states. This can be calculated approximately using quantum field theory, and so-called "the decay rate". Solve the differential equation to obtain

$$N(t) = N_0 \mathrm{e}^{-\Gamma t} \tag{4.6}$$

where N_0 is the initial radioactive nuclei number. With the definition of half-life, we have

$$\frac{1}{2}N_0 = N_0 \mathrm{e}^{-\Gamma T_{\frac{1}{2}}}$$
$$\Rightarrow$$
$$T_{\frac{1}{2}} = \frac{\ln 2}{\Gamma} \tag{4.7}$$

where we can see that the decay rate and the half-life have a linear relation.

Theory

The single beta theory at the nuclear level was first studied by Fermi in his article in 1934 [85]. The "Übergangswahrscheinlichkeit" (transition probability) depends on two parts separately multiplying together. One is the nuclear parts with respect to the initial and final states of neutrons and protons while the other refers to the electron/neutrino wave functions in relativistic situation (i.e. the solution of the wave function from the Dirac equation). The first is called *nuclear matrix element* (NME) and the latter is *phase space factor*. This theory of beta decay has been expanded in $\beta^-\beta^-$ decay with the consideration of the out-going leptons correlations. In 1935, Maria Göppert Mayer first proposed double beta decay [86]. Reviews of

Isotopes	$2\nu\beta\beta\ T_{\frac{1}{2}}(10^{21}yr)$	Method	$0\nu\beta\beta\ T_{\frac{1}{2}}(10^{23}yr)$	Method
^{48}Ca	$0.064^{+0.007}_{-0.006}$	NEMO-3	> 0.58	CaF_2 scint.
^{76}Ge	1.926 ± 0.094	GERDA	> 1800	GERDA
^{76}Ge			> 270	MAJORANA
^{76}Ge			> 157	Enriched HPGe
^{78}Kr	$9.2^{+5.5}_{-2.6}$	BAKSAN		
^{82}Se	0.096 ± 0.003	NEMO-3	> 24	CUPID-0
^{82}Se			> 1	NEMO-3
^{96}Zr	0.0235 ± 0.0014	NEMO-3		
^{100}Mo	0.00693 ± 0.00004	NEMO-3	> 0.95	AMoRE
^{100}Mo	$0.00712^{0.00018}_{0.00014}$	CUPID-Mo	> 11	NEMO-3
^{116}Cd	0.0274 ± 0.0004	NEMO-3	> 1	NEMO-3
^{116}Cd	$0.0263^{+0.0011}_{-0.0012}$	AURORA	> 2.2	AURORA
^{116}Cd	$0.029^{+0.004}_{-0.003}$	$CdWO_4$ scint.	> 1.7	$CdWO_4$ scint.
^{128}Te			> 1.1	Cryog. det.
^{130}Te	0.7 ± 0.09	NEMO-3	> 30	CUORICINO
^{130}Te	0.82 ± 0.02	CUORE-0		
^{134}Xe	> 0.87	EXO-200	> 1.1	EXO-200
^{136}Xe	2.38 ± 0.02	KamLAND-Zen	> 110	EXO-200
^{136}Xe	2.165 ± 0.016	EXO-200	> 2.4	PANDAX-II
^{136}Xe			> 2300	KamLAND-Zen
^{150}Nd	0.00934 ± 0.00022	NEMO-3		

Table 4.1: Isotopes of $\beta^-\beta^-$ decay and possible $0\nu\beta\beta$ process to 0^+ ground state. Data from [55, 82, 83]

the complete theory in recent 30 years are here [87, 88, 89]. We will follow [89] and write the brief derivations.

The beta decay can be described by the effective weak interaction Hamiltonian whose Feynman diagram is a four leg effective one.

$$H_{weak} = \frac{G_F \cos\theta_C}{\sqrt{2}}(j_{L\mu}J_L^{\mu\dagger} + \kappa j_{L\mu}J_R^{\mu\dagger} + \eta j_{R\mu}J_L^{\mu\dagger} + \lambda j_{R\mu}J_R^{\mu\dagger}) + h.c. \tag{4.8}$$

where G_F is the Fermi constant and $\cos\theta_C$ is the Cabibbo-Kobayashi-Maskawa angle. j, J are leptonic and hadronic currents, respectively. The coupling constants κ, η, λ are

$$\lambda = \frac{(\frac{m_{W_1}}{m_{W_2}})^2 + \tan^2\zeta}{1 + (\frac{m_{W_1}}{m_{W_2}})^2 \tan^2\zeta}; \qquad \eta = \kappa = -\frac{[1 - (\frac{m_{W_1}}{m_{W_2}})^2]\tan\zeta}{1 + (\frac{m_{W_1}}{m_{W_2}})\tan^2\zeta} \tag{4.9}$$

where m_{W_1}, m_{W_2} are mass eigenstates similar with (3.41) and $\zeta = \xi_W$ is the mixing angle in the fixed right-handed current model. If we only consider the left-handed current in the $2\nu\beta\beta$ process, the differential decay rate is

$$d\Gamma_{2\nu} = 2\pi \sum_{i,j} \sum_{spin} |\mathcal{A}_{2\nu}|^2 \delta(\epsilon_1 + \epsilon_2 + w_1 + w_2 + E_F - E_I) \frac{d\vec{p}_1}{(2\pi)^3} \frac{d\vec{p}_2}{(2\pi)^3} \frac{d\vec{k}_1}{(2\pi)^3} \frac{d\vec{k}_2}{(2\pi)^3} \tag{4.10}$$

where $\mathcal{A}$ is the matrix element amplitude. $p_1, p_2, \epsilon_1, \epsilon_2$ are the momenta and energies of the emitting electrons. w_1, w_2 are the energies of the emitting neutrinos. If we only consider the S-wave leptons, the combinations of the angular momentum permit two transition processes: $0^+ \to 0^+$ and $0^+ \to 2^+$. The decay rate can be expressed as leptonic parts and the nuclear interaction parts. With Equation (4.7) we have

$$(T^{2\nu}_{\frac{1}{2}})^{-1} = \frac{\Gamma_{2\nu}}{\ln 2} = \frac{F_{2\nu} M_{2\nu}}{\ln 2} \tag{4.11}$$

where $F_{2\nu}$ and $M_{2\nu}$ are phase space factor and the NME. One can find the detail derivations of the phase space factor in a review paper [90]. They also numerically solved the radial wave function of electrons from the Dirac equation under different assumptions of the nuclear potential. The codes are in another article [91]. We only show an overview of the $2\nu\beta\beta$ decay. We will calculate the $0\nu\beta\beta$ decay rate under mLRSM in the following sections.

4.2 The Standard Mechanism

In the double beta decay, the neutrinos and electrons are supposed to emit out. The process can be described as two single beta decays with the correlation of the out-going leptons. Thus, the diagram in this way looks like setting two single beta decay together. What if the neutrinos in Figure 4.2 are Majorana particles? The Majorana particle is its own antiparticle so there will be a neutrino exchange in the mediate that connects the two single beta decay diagrams (Figure 4.3). This means that the neutrinos can propagate in the neutrinoless double decay and there will be no out-going neutrino. The $0\nu\beta\beta$ process with two out-going electrons is

$$(A, Z) \longrightarrow (A, Z + 2) + 2e^- \tag{4.12}$$

We can use (4.4) to rewrite the other types of processes for $0\nu\beta\beta$. Since we do not know whether the neutrinos are Majorana or Dirac type, any isotopes observed in double beta decay can be candidates in the neutrinoless double decay, see right panels of Table 4.1. If the neutrinoless double beta decay rate is not much smaller than the corresponding double beta decay rate, $0\nu\beta\beta$ is not suppressed and the detection is possible. The standard mechanism of neutrinoless double decay refers to the possible decay under the SM symmetry group with the Majorana neutrino fixing. In Chapter 2, we add the right-handed singlet to form Majorana neutrinos.

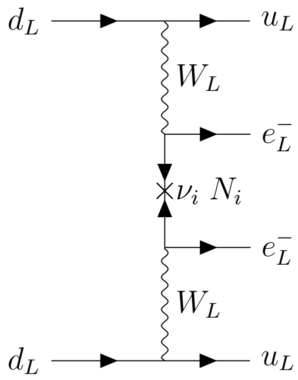


Figure 4.3: The 'standard mechanism' with two left-handed W bosons

In the standard mechanism, only the left-handed currents are taken into account. The diagram shows in Figure 4.3, where we consider the light neutrino propagating in the mediate state since we only have left-handed lepton-boson interaction in the standard mechanism. One needs to notice that the neutrinoless process is definitely not a SM process, so the standard mechanism does not equal to the SM mechanism. Historically, the terminology may come from the fact that the theory of double beta decay and neutrinoless double decay appeared earlier than LRSM or SUSY [11, 86, 92]. In the past half century, the theories have been studied by several papers. For a review of the recent study, the readers can find here: [54, 93].

We are interested in the phase space factor and the NME, and we need to calculate the decay rate. The process of calculating the decay rate is nothing different from the calculation of physical expectation value, or physical observable multiplied with the density of states. We consider the low-energy non-thermal decay process, where we sandwich the Hamiltonian between the initial and final states and integrate over the whole space. The four dimension momentum is conserved in the process. Explicitly, we should be aware of the different contribution in each diagram from different channels. This comes from the exchange of the momenta of the out-going electrons, while in the non-standard mechanism, this comes from the exchange of chiralities in each vertex as well. One finds the specific details in every single calculation of the diagram. The general expression of differential decay rate in Fermi's golden rule is [94]

$$d\Gamma = \frac{(2\pi)^4}{2E_i}|\mathcal{M}|^2\delta^4(\sum_{i,f}(p_f - p_i))\prod_f \frac{d^3p_f}{(2\pi)^3 2E_f} \tag{4.13}$$

where i, f sum over all the initial states and final states, respectively. $\mathcal{M}$ is the Lorentz-invariant matrix element, or the matrix element. Its physical meaning is the Lorentz-invariant transition matrix from all initial states to final states. The matrix element $\mathcal{M}$ can be expressed as

$$|\mathcal{M}|^2 = (\prod_{i,f} 2E_i 2E_f)|\langle f|\,\mathcal{A}\,|i\rangle|^2 \tag{4.14}$$

where $\mathcal{A}$ is the amplitude operator of the process. Now, let us first find $\mathcal{A}$. In a scattering process, the matrix sandwiched in the initial and final states is usually called the S-matrix (scattering matrix). This is the correlation function including the in-going and out-going particles wave functions, propagators, and momentum conservation couplings (vertices). In order to make our life easier, we can directly write down it through Feynman diagrams 4.3 in the tree level approximation by using the Feynman rules (see Appendix E).

In literature, people are mostly using the effective field theory [95]. With some reasonable approximations , specific Lagrangian can be reduced to an effective Lagrangian which helps the further analytic calculations. The effective Lagrangian are not perturbatively renormalizable at most of the time, because the couplings usually have negative mass dimensions. For example, the famous 4-fermion interact picture in dimension-6 theory is not renormalizable. This problem cannot be avoided, but one can add fixed couplings and a manual cutoff [96, 97]. When we are talking about the electroweak theory of the SM, the heavy W/Z bosons naturally give this cutoff due to their mass scale in the long-range approximation. The mass scales of different bosons are related to the symmetry of the model, and this is one of the reasons why people are building models to find the solution of the hierarchy problem. In other words, it is efficient and mathematically safe to use an effective Lagrangian. Let us start with the standard case with two left-handed gauge bosons, see Figure 4.3. The amplitude of this diagram is

$$\begin{aligned}\mathcal{A} = {} & \overline{u}(\vec{p_2})(-\frac{i}{\sqrt{2}}\gamma^\mu g_L P_L V_{ud}^*)d(\vec{p_1}) \cdot \frac{ig_{\mu\nu}}{p_{W_L}^2 - m_{W_L}^2 + i\epsilon} \\ & \cdot \left[\overline{e}^u(\vec{p_{e1}})(-\frac{i}{\sqrt{2}}g_L\gamma^\nu P_L)\right] \cdot P_L \frac{-iU_{ei}^2(\not{p} + m_{\nu i})}{p^2 - m_\nu^2 + i\epsilon} P_L C \cdot \left[\overline{e}^u(\vec{p_{e2}})(-\frac{i}{\sqrt{2}}g_L\gamma^\rho P_L)\right] \\ & \cdot \frac{ig_{\rho\sigma}}{p_{W_L}^2 - m_{W_L}^2 + i\epsilon} \cdot \overline{u}(\vec{p_4})(-\frac{i}{\sqrt{2}}\gamma^\sigma g_L P_L V_{ud})d(\vec{p_3})\end{aligned} \tag{4.15}$$

where $u(p_{2,4}), d(p_{1,3})$ are the quark wave functions and $e^u(p_{e1,e2})$ is the electron wave function with one of the spinor components u. $\not{p} = \gamma^\mu p_\mu$ and p in the propagator is the neutrino momentum. m_ν is the neutrino mass (the subscripts in m are not Lorentz indices. I have to again apologize for the inconvenience sub/superscripts. We are running out of Greek letters, and this funny phenomenon sometimes happens in studying physics.). The charge conjugate operator can be eliminate by

$$\begin{aligned} C\left[\overline{u}\gamma^\mu P_L\right]^T &= CP_L^T\gamma^{\mu T}\overline{u}^T = P_L\gamma^\mu C\overline{u}^T = \gamma^\mu P_R u^c \\ &= \gamma^\mu P_R v \end{aligned} \tag{4.16}$$

where v is the other spinor. We can also use $P_L P_L = P_L$ and the position exchange of the gamma matrices, then Equation (4.15) is reduced to

$$\mathcal{A} = (-ig_L)^4 \cdot \frac{1}{4} \cdot \overline{u}(\vec{p_2})\gamma^\mu P_L V_{ud}^* d(\vec{p_1}) \cdot \frac{ig_{\mu\nu}}{p_{W_L}^2 - m_{W_L}^2 + i\epsilon} \cdot \overline{e^u}(\vec{p_{e1}})(\gamma^\nu P_L)\times$$
$$\frac{-i(\not{p} + U_{ei}^2 m_{\nu i})}{p^2 - m_\nu^2 + i\epsilon} \cdot (P_L\gamma^\rho)(e^u)^c(\vec{p_{e2}}) \cdot \frac{ig_{\rho\sigma}}{p_{W_L}^2 - m_{W_L}^2 + i\epsilon} \cdot \overline{u}(\vec{p_4})\gamma^\sigma P_L V_{ud} d(\vec{p_3}) \quad (4.17)$$

If we consider the propagating energy of the process as 100MeV scale, it is much smaller than the mass scale of the left-handed W boson which is known as 80GeV. That is to say, when we are dealing with heavy particle propagators, the mass part dominates. The momentum part can be ignored by using the method so-called the "integrate out". The idea is to integrate out the large scale Λ field from the path integral of the generating function. We will obtain a effective action that only depends on the rest field but not on the heavy field, and the effect of the heavy field is represented in the derivative expansions ordered in the power series of the large energy scale Λ. The derivations are in the Appendix C. After "integrating out" the W bosons, we have

$$\mathcal{A}_{eff} == -i\frac{g_L^4}{32m_{W_L}^2}V_{ud}^2 \cdot \overline{e^u}(\vec{p_{e1}})\gamma_\mu \frac{U_{ei}^2 m_{\nu i}}{p^2 - m_\nu^2 + i\epsilon}(1-\gamma^5)\gamma_\rho(e^u)^c(\vec{p_{e2}}) \cdot J_L^\mu J_L^\rho \quad (4.18)$$

$\not{p}$ disappears since P_L changes into P_R when passing through odd number of gamma matrices, see Equations (4.33) and (4.35). In the last equation we use (E.18) to define the hadronic current J

$$\begin{aligned} \overline{u}(\vec{p_2})\gamma^\mu(1-\gamma^5)d(\vec{p_1}) &\equiv J_L^\mu(\vec{p_2}, \vec{p_1}) \\ \overline{u}(\vec{p_4})\gamma^\rho(1-\gamma^5)d(\vec{p_3}) &\equiv J_L^\rho(\vec{p_4}, \vec{p_3}) \end{aligned} \quad (4.19)$$

the effective neutrino mass is much smaller than the possible propagating neutrino momentum p, thus the momentum dominates in the denominator. Finally we obtain the effective amplitude

$$\mathcal{A}_{eff}^{(LL)} = iG_F^2(U_{ei}^2 m_{\nu i})V_{ud}^2 \cdot \overline{e^u}(\vec{p_{e1}})\gamma_\mu \frac{1}{p^2}(1-\gamma^5)\gamma_\rho(e^u)^c(\vec{p_{e2}}) \cdot J_L^\mu J_L^\rho \quad (4.20)$$

where G_F is the Fermi constant, $G_F = \sqrt{2}g^2/(8m_{W_L}^2)$. From Equation (4.20) we know that the amplitude of the lowest level of the neutrinoless double decay is proportional to the mass scale of the neutrino mass. However, this is not very physical, unless we insert the amplitude into the initial and final states to calculate the cross section or rate. In the position basis, the propagator contains the integration of full momentum space which is part of the NME.

Let us first calculate the "standard mechanism" matrix element. From now on, we will use the abbreviations LL for the diagram 4.3 and RR, LR-λ, LR-η LH, RH for the diagrams from 4.5 to 4.9. The matrix element of the standard mechanism is

$$\begin{aligned}
&\langle f| \mathcal{A}_{eff}^{(LL)} |i\rangle \\
=&iG_F^2(U_{ei}^2 m_{\nu i})V_{ud}^2 N_{p_1} \int d^4x_1 \overline{u^s}(p_1)\mathrm{e}^{ip_1\cdot x_1}\gamma_\mu \\
&\times \frac{1}{(2\pi)^4}\int d^4p \frac{\mathrm{e}^{-ip\cdot(x_1-x_2)}}{p^2}(1-\gamma^5)\gamma_\rho N_{p_2}\int d^4x_2 v^{s'}(\vec{p_2})\mathrm{e}^{ip_2\cdot x_2}\cdot \langle f| T(J_L^\mu J_L^\rho) |i\rangle \\
&-(p_1 \longleftrightarrow p_2)
\end{aligned} \tag{4.21}$$

where we use the shorthand writing $p_1 \equiv p_{e1}, p_2 \equiv p_{e2}$. We suppose that the electron wave functions as plane wave package approximation with spinors $u^s, v^{s'}$, and s, s' stand for the spin states. N_{p_1} and N_{p_2} are normalization factor of the electron wave package functions, $N_{p_{1,2}} = \frac{1}{(2\pi)^{\frac{3}{2}}\sqrt{p^0_{1,2}}}$, and p^0 is the energy component of the 4-dimensional momentum. The neutrino propagator is expanded in the position basis. Use residue theorem to integrate the time component, and take into account $x_1^0 > x_2^0$.

$$\lim_{\epsilon\to 0}\frac{1}{(2\pi)^4}\int d^4p \frac{\mathrm{e}^{-ip\cdot(x_1-x_2)}}{(p^0)^2-m^2+i\epsilon} = \frac{1}{(2\pi)^3}\int d^3p \frac{\mathrm{e}^{-ip^0(x_1^0-x_2^0)}\mathrm{e}^{\vec{p}\cdot(\vec{x_1}-\vec{x_2})}}{2p^0} \tag{4.22}$$

For the other time ordered case, we will have $x_2^0 - x_1^0$ in the nominator. $T(J_L^\nu J_L^\rho)$ is the time ordering of the hadronic current. The last line of (4.21) means the correspondent matrix element with the momentum exchange of out-going electrons. Since the correlation function can be solved perturbatively by expansion series of Hamiltonian in the effective theory, we adjust to the second order perturbation and introduce the mediate states by inserting the identity operator $\sum_n |N_n\rangle\langle N_n| = \mathbb{1}$.

$$\begin{aligned}
\langle f| T(J_L^\nu J_L^\rho) |i\rangle = \sum_n \big[&\langle f| J_L^\nu(x_1)|N_n\rangle\langle N_n| J_L^\rho(x_2)|i\rangle \\
&+ \langle f| J_L^\nu(x_2)|N_n\rangle\langle N_n| J_L^\rho(x_1)|i\rangle \big]
\end{aligned} \tag{4.23}$$

and we can extract the time component by using the time evolution in Heisenberg picture.

$$\begin{aligned}
\langle f| T(J_L^\nu J_L^\rho) |i\rangle = \sum_n \big[&\mathrm{e}^{i(E_f-E_n)x_1^0}\mathrm{e}^{i(E_n-E_i)x_2^0}\langle f| J_L^\nu(\vec{x}_1)|N_n\rangle\langle N_n| J_L^\rho(\vec{x}_2)|i\rangle \\
&+ \mathrm{e}^{i(E_f-E_n)x_2^0}\mathrm{e}^{i(E_n-E_i)x_1^0}\langle f| J_L^\nu(\vec{x}_2)|N_n\rangle\langle N_n| J_L^\rho(\vec{x}_1)|i\rangle \big]
\end{aligned} \tag{4.24}$$

E_n are mediate energies. x^0 is the time component, and one can use the variables separation to draw out the time part. Combine the electron package and neutrino propagator, the energy integration is

$$\int_{-\infty}^{\infty} dx_1^0 dx_2^0 \mathrm{e}^{ip_1^0 x_1^0} \mathrm{e}^{-ip^0(x_1^0-x_2^0)} \mathrm{e}^{ip_2^0 x_2^0} \left[\mathrm{e}^{i(E_f-E_n)x_1^0} \mathrm{e}^{i(E_n-E_i)x_2^0} + \mathrm{e}^{i(E_f-E_n)x_2^0} \mathrm{e}^{i(E_n-E_i)x_1^0}\right] \tag{4.25}$$

Since we have the time ordering, the integration range of (4.25) can be separated to match the each ordering.

$$\begin{aligned} &\int_{-\infty}^{\infty} dx_1^0 \int_{-\infty}^{x_1^0} dx_2^0 \mathrm{e}^{i(p_1^0-p^0+E_f-E_n)x_1^0} \mathrm{e}^{i(p_2^0+p^0+E_n-E_i-i\epsilon)x_2^0} \\ &+ \int_{-\infty}^{\infty} dx_2^0 \int_{-\infty}^{x_2^0} dx_1^0 \mathrm{e}^{i(p_1^0-p^0+E_n-E_i-i\epsilon)x_1^0} \mathrm{e}^{i(p_2^0+p^0+E_f-E_n)x_2^0} \end{aligned} \tag{4.26}$$

where in the first term we have $x_1^0 > x_2^0$, while $x_2^0 > x_1^0$ in the second term. The final result of the integral is

$$\left(\frac{-i}{E_n + p_2^0 + p^0 - E_i - i\epsilon} + \frac{-i}{E_n + p_1^0 + p^0 - E_i - i\epsilon}\right) \cdot 2\pi\delta(E_f + p_1^0 + p_2^0 - E_i) \tag{4.27}$$

where we use the formula below and the Dirac delta function,

$$\begin{aligned} &\int_0^{\infty} dx^0 \mathrm{e}^{i(a+i\epsilon)x^0} = \lim_{\epsilon \longrightarrow 0} \frac{i}{a+i\epsilon} \qquad \int_{-\infty}^{0} dx^0 \mathrm{e}^{i(a-i\epsilon)x^0} = \lim_{\epsilon \longrightarrow 0} \frac{-i}{a-i\epsilon} \\ &\int_{-\infty}^{y^0} dx^0 \mathrm{e}^{i(a-i\epsilon)x^0} = \lim_{\epsilon \longrightarrow 0} \frac{-i}{a-i\epsilon} \mathrm{e}^{i(a-i\epsilon)y^0} \end{aligned} \tag{4.28}$$

Combine Equation (4.22)-(4.28) to obtain the matrix element,

$$\begin{aligned} &\langle f | \mathcal{A}_{eff}^{(LL)} | i \rangle \\ =&iG_F^2 (\sum_i^3 U_{ei}^2 m_{\nu_i}) V_{ud}^2 \cdot N_{p_1} N_{p_2} \overline{u^s}(p_1) \gamma_\mu (1-\gamma^5) \gamma_\rho v^{s'}(p_2) \\ &\times \int d^3x_1 d^3x_2 \mathrm{e}^{-i\vec{p}_1\cdot\vec{x}_1 - i\vec{p}_2\cdot\vec{x}_2} \cdot \frac{1}{(2\pi)^3} \int d^3p \frac{\mathrm{e}^{-i\vec{p}\cdot(\vec{x}_1-\vec{x}_2)}}{2p^0} \\ &\times \Big[\sum_n \frac{\langle f | J_L^\mu(x_1) | N_n \rangle \langle N_n | J_L^\rho(x_2) | i \rangle}{E_n + p_2^0 + p^0 - E_i - i\epsilon} + \frac{\langle f | J_L^\mu(x_2) | N_n \rangle \langle N_n | J_L^\rho(x_1) | i \rangle}{E_n + p_1^0 + p^0 - E_i - i\epsilon}\Big] \\ &\times 2\pi\delta(E_f + p_1^0 + p_2^0 - E_i) - (p_1 \longleftrightarrow p_2) \end{aligned} \tag{4.29}$$

It is not easy to calculate (4.29) further analytically, unless we discuss some approximations [36].
1) Small neutrino mass: $p^0 = \sqrt{\vec{p}^2 + m_i^2} \approx \vec{p}$.

2) Long wave approximation that two electrons are supposed to emit only in S-states $e^{-i\vec{p}_1\cdot\vec{x}_1-i\vec{p}_2\cdot\vec{x}_2}\approx 1$.
3) Closure approximation: introduce the effective energy of the middle state $\overline{E_n}$. The middle states are considered effectively.
4) Neglect the kinetic energy of the recoil nucleus in the laboratory frame: $M_i = M_f + p_1^0 + p_2^0 \to p_{1,2} + p + \overline{E_n} - M_i = \frac{p_1^0-p_2^0}{2} + p + \overline{E_n} - \frac{M_i+M_f}{2} \approx p + \overline{E_n} - \frac{M_i+M_f}{2}$.
5) The impulse approximation: $\vec{J}_L^\mu(\vec{x}_1)\vec{J}_L^\nu(\vec{x}_2) = \vec{J}_L^\mu(\vec{x}_2)\vec{J}_L^\nu(\vec{x}_1)$. And in this approximation we have [89] $J_L^\mu(\vec{x}) \approx \sum_n \delta(\vec{x}-\vec{r}_n)\tau_+^n[g_V(q^2)g^{\mu 0} + g_A(q^2)\sigma_i^n g^{\mu i}]$, where $\tau_+ = \frac{1}{2}(\tau_1 + i\tau_2)$. τ_i and σ_i are Pauli matrices. τ_+ converts a neutron state to a proton state. Two left-handed hadronic currents correlation is (contract the indices of gamma matrices)

$$
\begin{aligned}
&\vec{J}_{\mu L}(\vec{x}_1)\vec{J}_L^\mu(\vec{x}_2)\\
&=\sum_{n,m}\delta(\vec{x}_1-\vec{r}_m)\delta(\vec{x}_2-\vec{r}_n)\tau_+^n\tau_+^m[g_V(q^2)g^{\mu 0}+g_A(q^2)\sigma_i^n g^{\mu i}][g_V(q^2)g_{\mu 0}+g_A(q^2)\sigma_i^m g_{\mu i}]\\
&=\sum_{n,m}\delta(\vec{x}_1-\vec{r}_n)\delta(\vec{x}_2-\vec{r}_m)\tau_+^n\tau_+^m[g_V^2-(\vec{\sigma}^n\cdot\vec{\sigma}^m)g_A^2] \qquad (4.30)
\end{aligned}
$$

With all approximations and Equation (4.30), the matrix element has the form

$$
\begin{aligned}
\langle f|\,\mathcal{A}_{eff}^{(LL)}\,|i\rangle =& iG_F^2(\sum_i^3 U_{ei}^2 m_{\nu i})V_{ud}^2\cdot N_{p_1}N_{p_2}\overline{u^s}(p_1)(1+\gamma^5)v^{s'}(p_2)\\
&\times\sum_{n,m}\langle N_f|\,\frac{1}{(2\pi)^3}\int d^3p\frac{e^{i\vec{p}\cdot(\vec{r}_n-\vec{r}_m)}}{\vec{p}(\vec{p}+\overline{E_n}-\frac{M_i+M_f}{2})}\tau_+^n\tau_+^m[g_V^2-\vec{\sigma}^n\cdot\vec{\sigma}^m g_A^2]\,|N_i\rangle\\
&\times\pi\delta(E_f+p_1^0+p_2^0-E_i)-(p_1\longleftrightarrow p_2) \qquad (4.31)
\end{aligned}
$$

where we use the Clifford algebra:

$$
\gamma_\mu\gamma_\rho = g_{\mu\rho} + \frac{1}{2}(\gamma_\mu\gamma_\rho - \gamma_\rho\gamma_\mu) \qquad (4.32)
$$

And only the first term of (4.32) contributes. The second term vanishes when contracting with the hadronic currents under the impulse approximation. The exchange of momentum in the LL case will lead to a factor 2 [89]. Let us generally do it step by step. Exchange the momentum and do transpose, with the neutrino propagator vector an additional a minus sign for inverse propagating $\vec{p}\Rightarrow -\vec{p}$

$$
\begin{aligned}
&(\overline{e}(p_2)P_\alpha(\frac{\gamma_0p^0+\vec{\gamma}\cdot\vec{p}+m_\nu}{p^2-m^2})P_\beta e^c(p_1))^T = \overline{e}(p_1)C^TP_\beta^T(\frac{\gamma_0p^0+\vec{\gamma}\cdot\vec{p}+m_\nu}{p^2-m^2})P_\alpha^T\overline{e}^T(p_2)\\
&=-\overline{e}(p_1)P_\beta(\frac{-\gamma_0p^0-\vec{\gamma}\cdot\vec{p}+m_\nu}{p^2-m^2})P_\alpha e^c(p_2) \qquad (4.33)
\end{aligned}
$$

where $P_{\alpha,\beta}$ is the projection operator, α,β can be taken left (L) or right (R). In the

LL (also in RR) case, the same projection operators will eliminate the momentum term in the nominator. With the original term, we have

$$\overline{e}(p_1)P_\beta(\frac{\gamma_0 p^0 - \vec{\gamma}\cdot\vec{p} + m_\nu}{p^2 - m^2} - \frac{\gamma_0 p^0 + \vec{\gamma}\cdot\vec{p} - m_\nu}{p^2 - m^2})P_\alpha e^c(p_2) \tag{4.34}$$

$$\xrightarrow{\alpha,\beta=L}$$

$$2\overline{e}(p_1)\frac{m_\nu}{p^2 - m^2}P_L e^c(p_2) \tag{4.35}$$

where we obtain the factor 2. It is obvious that the result will be totally different if we choose LR chirality. We will discuss this in the LR cases. Next, calculate the integration in (4.31) with respect to the neutrino momentum in the spherical coordinates to further simplify the matrix element.

$$\begin{aligned}
&\frac{1}{(2\pi)^3}\int d^3p \frac{\mathrm{e}^{i\vec{p}\cdot(\vec{r}_n - \vec{r}_m)}}{\vec{p}(\vec{p} + \overline{E_n} - \frac{M_i+M_f}{2})} \\
=&\frac{1}{(2\pi)^3}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^{\infty} dp p^2 \sin\theta \frac{\mathrm{e}^{i\vec{p}\cdot(\vec{r}_n - \vec{r}_m)}}{\vec{p}(\vec{p} + \overline{E_n} - \frac{M_i+M_f}{2})} \\
=&\frac{1}{(2\pi)^3}\int_0^{2\pi} d\phi \int_0^{\infty} dp p^2 \frac{1}{\vec{p}(\vec{p} + \overline{E_n} - \frac{M_i+M_f}{2})}\int_0^{\pi} d\theta \sin\theta \mathrm{e}^{i(|p||r_{nm}|\cos\theta)}
\end{aligned} \tag{4.36}$$

use Euler's identity and integrate the two angles,

$$\begin{aligned}
=&\frac{1}{(2\pi)^2}\int_0^{\infty} dp p^2 \frac{1}{|p||r_{nm}|}\frac{2\sin|p||r_{nm}|}{\vec{p}(\vec{p} + \overline{E_n} - \frac{M_i+M_f}{2})} = \frac{1}{2(\pi)^2 r_{nm}}\int_0^{\infty} dp \frac{\sin(p r_{nm})}{(\vec{p} + \overline{E_n} - \frac{M_i+M_f}{2})} \\
\equiv&\frac{1}{4\pi R}H(r_{nm}, \overline{E_n})
\end{aligned} \tag{4.37}$$

where we define the neutrino potential function:

$$H(r_{nm}, \overline{E_n}) = \frac{2R}{\pi r_{nm}}\int_0^{\infty} dp \frac{\sin(p r_{nm})}{p + \overline{E_n} - \frac{M_i+M_f}{2}} \tag{4.38}$$

and $r_{nm} \equiv r_n - r_m$, p stands for $|\vec{p}|$. Then the matrix element (4.31) becomes

$$\begin{aligned}
\langle f|\,\mathcal{A}_{eff}^{(LL)}\,|i\rangle =&2iG_F^2(\sum_i^3 U_{ei}^2 m_{\nu i})V_{ud}^2 \cdot N_{p_1}N_{p_2}\overline{u^s}(p_1)(1+\gamma^5)v^{s'}(p_2) \\
&\times \sum_{n,m}\langle N_f|\,\frac{1}{4\pi R}H(r_{nm}, \overline{E_n})\tau_+^n\tau_+^m[g_V^2 - \vec{\sigma}^n\cdot\vec{\sigma}^m g_A^2]\,|N_i\rangle \\
&\times \pi\delta(E_f + p_1^0 + p_2^0 - E_i)
\end{aligned} \tag{4.39}$$

Define the so-called NME in LL:

$$\begin{aligned} M_{0v}^{(LL)} &= M_{0v,GT}^{(LL)} - \frac{g_V^2}{g_A^2} M_{0v,F}^{(LL)} \\ M_{0v,GT}^{(LL)} &\equiv \sum_{n,m} \langle N_f| \, H(r_{nm}, \overline{E_n}) \tau_+^n \tau_+^m \vec{\sigma}^n \cdot \vec{\sigma}^m \, |N_i\rangle \\ M_{0v,F}^{(LL)} &\equiv \sum_{n,m} \langle N_f| \, H(r_{nm}, \overline{E_n}) \tau_+^n \tau_+^m \, |N_i\rangle \end{aligned} \tag{4.40}$$

The superscript identifies different cases, here we use superscript (LL) for the LL case. The subscripts GT and F stand for the Gamow-Teller matrix element and the Fermi matrix element, respectively [84]. $g_V(q^2), g_A(q^2)$ are vector and axial vector form factors [94], we have [89]

$$g_V(q^2 = 0) = 1 \tag{4.41}$$

$$g_A(q^2 = 0) = 1.254 \tag{4.42}$$

where q^2 is the transfer momentum. Using these definitions, the matrix element becomes

$$\begin{aligned} \langle f| \, \mathcal{A}_{eff}^{(LL)} \, |i\rangle =& iG_F^2 (\sum_i^3 U_{ei}^2 m_{\nu i}) V_{ud}^2 \cdot N_{p_1} N_{p_2} \overline{u^s}(p_1)(1+\gamma^5) v^{s'}(p_2) \frac{g_A^2}{2R} M_{0\nu}^{(LL)} \\ & \times \delta(E_f + p_1^0 + p_2^0 - E_i) \end{aligned} \tag{4.43}$$

Then we can do the absolute square of the matrix element and sum over the spin states of the out-going electrons. The sum over of the spins behaves as the trace. Add the spinor indices to show it explicitly

$$\begin{aligned} & \sum_{s,s'} (\overline{u^s}_a(p_1)(1+\gamma^5)_{ab} v_b^{s'}(p_2))^\dagger \overline{u^s}_a(p_1)(1+\gamma^5)_{ab} v_b^{s'}(p_2) \\ =& \sum_{s,s'} \overline{v}_c^{s'}(p_2)(1-\gamma^5)_{cd} u_d^s(p_1) \overline{u^s}_a(p_1)(1+\gamma^5)_{ab} v_b^{s'}(p_2) \end{aligned} \tag{4.44}$$

using the spinors' completeness relations (E.12) and calculating the trace, we will obtain

$$\frac{1}{4} \mathrm{Tr}[(\not{p}_2 - m_e)(1-\gamma^5)(\not{p}_1 + m_e)(1+\gamma^5)] = 2 p_1 \cdot p_2 \tag{4.45}$$

The full derivations of the trace is in Appendix F. The dot product of 4-dimensional momenta can also be expressed as

$$p_1 \cdot p_2 = E_1 E_2 - p_1 p_2 \cos\xi \tag{4.46}$$

where E_1, E_2 are the energy of the out-going electrons. Factor $\frac{1}{4}$ exists when detectors cannot distinguish the electron polarization. The leptonic part contains the electron correlated angle ξ, which is an important observable in experiments. With (4.45) and 4.46, the square of the matrix element is

$$\begin{aligned}\left|\langle f|\,\mathcal{A}_{eff}^{(LL)}\,|i\rangle\right|^2 =& (\sum_i^3 U_{ei}^2 m_{\nu i})^2 \frac{G_F^4 V_{ud}^4 g_A^4}{2(2\pi)^6 E_1 E_2 R^2}(E_1E_2 - p_1p_2\cos\xi)\cdot\left|M_{0\nu}^{(LL)}\right|^2 \\ &\cdot\delta(E_f + p_1^0 + p_2^0 - E_i)\end{aligned} \tag{4.47}$$

One more necessity needs to be considered. The out-going electrons in the decay process will interact with the Coulomb potential of the nucleus. This attractive electromagnetic force gives a screening effect to the electrons. In the non-relativistic approximation, we use the Fermi approximation $\frac{1}{1-\mathrm{e}^{-2\pi\eta}}$ to approach the effective nuclear potential. This is done by multiplying the Fermi function with each electron wave function. The Fermi function is

$$F(Z,E) = \frac{2\pi\eta}{1-\mathrm{e}^{-2\pi\eta}} \tag{4.48}$$

where $\eta = \frac{Z\alpha m_e}{p_{(1,2)}}$, Z is the proton number of the mother nucleus and $\alpha = 1/137$ is the Sommerfeld constant or the fine structure constant. To compensate for the momentum in η, we need to multiply p_1p_2 in the formula. Inserting all these back into the general differential equation (4.13) and considering the momenta differential $p_1^2p_2^2\sin\xi d\xi dp_2$, we obtain

$$\begin{aligned}d\Gamma_{(LL)}^{0\nu} =& |m_{\beta\beta}|^2\left|M_{0\nu}^{(LL)}\right|^2 \frac{G_F^4 V_{ud}^4 g_A^4}{(2\pi)^5 R^2}(E_1E_2 - p_1p_2\cos\xi) \\ &\times F(Z+2,E_1)F(Z+2,E_2)p_1p_2\sin\xi d\xi dE_2\end{aligned} \tag{4.49}$$

where $|m_{\beta\beta}|^2 \equiv (\sum_i^3 U_{ei}^2 m_{\nu i})^2$. The half-life in Equation (4.11) can be expressed as the integrate of (4.49),

$$(T_{\frac{1}{2}})^{-1} = \frac{\Gamma_{(LL)}^{0\nu}}{\ln 2} = |m_{\beta\beta}|^2\left|M_{0\nu}^{(LL)}\right|^2 G_{0\nu}^{(LL)}(Q,Z) \tag{4.50}$$

where the phase space factor $G_{(LL)}^{0\nu}$ is

$$\begin{aligned}G_{(LL)}^{0\nu} =& \frac{G_F^4 V_{ud}^4 g_A^4}{2\ln 2(2\pi)^5 R^2}\int_0^Q dT_1 \int_0^\pi \sin\xi d\xi (E_1E_2 - p_1p_2\cos\xi)p_1p_2 \\ &\times F(Z+2,E_1)F(Z+2,E_2)\end{aligned} \tag{4.51}$$

where $T_1 = E_1 - m_e,\ Q = M_i - M_f - 2m_e,\ Q$ is the total decay energy (kinetic energy released in the decay process). The additional factor $\frac{1}{2}$ refers to two identical out-going electrons in the final states.

4.2.1 Effective Neutrino Mass

As we have already mentioned, the flavor states of neutrinos are

$$\nu_\alpha = \sum_i U^*_{\alpha i}\nu_i \tag{4.52}$$

where $U \equiv U_{majorana} = U^*_\alpha$ is the Majorana transformation. Instead of the standard parametrization in 2.5, we will use symmetrical parametrization [98] where each rotation gains a phase. That is to say, the Majorana phase is not of the form of (2.87), but additional phases $\phi_{12}, \phi_{13}, \phi_{23}$ are added to the corresponding rotation in the first line of (2.83). In this way, we have for U [99]

$$U = \begin{pmatrix} c_{13}c_{12} & c_{13}s_{12}\mathrm{e}^{-i\phi_{12}} & s_{13}\mathrm{e}^{-i\phi_{13}} \\ -s_{23}s_{13}c_{12}\mathrm{e}^{-i(\phi_{23}-\phi_{13})} - c_{23}s_{12}\mathrm{e}^{i\phi_{12}} & -s_{23}s_{13}s_{12}\mathrm{e}^{-i(\phi_{12}+\phi_{23}-\phi_{13})} + c_{12}c_{23} & s_{23}c_{13}\mathrm{e}^{-i\phi_{23}} \\ -c_{23}s_{13}c_{12}\mathrm{e}^{i\phi_{13}} + s_{23}s_{12}\mathrm{e}^{i(\phi_{12}+\phi_{23})} & -c_{23}s_{13}s_{12}\mathrm{e}^{-i(\phi_{12}-\phi_{13})} - s_{23}c_{12}\mathrm{e}^{i\phi_{23}} & c_{23}c_{13} \end{pmatrix} \tag{4.53}$$

where the Majorana phase indicates the CP violation. The neutrino mass transformed between the flavor and mass eigenstates is proportional to U^2 from Equation (4.52). Mathematically, if we want to transform to mass eigenstates in mass terms, e.g. $\overline{\nu}\nu$, there will be U^2 afterward. We have already used it in the derivation for the differential rate in the last section. The electron neutrino mass in $0\nu\beta\beta$ with the linear combination of mass eigenstates will be

$$\langle m_{ee}\rangle \equiv \langle m_{\beta\beta}\rangle = \left|\sum_i U^2_{ei} m_i\right| \tag{4.54}$$

with the matrix (4.53), we can write

$$\langle m_{ee}\rangle = \left|c^2_{12}c^2_{13}m_1 + s^2_{12}c^2_{13}m_2\mathrm{e}^{2i\phi_{12}} + s^2_{13}m_3\mathrm{e}^{2i\phi_{13}}\right| \tag{4.55}$$

From the neutrino oscillation (section 2.5), we know that masses $m_{1,2,3}$ cannot be directly detected from experiments. The oscillation experiments only provide the mass scale of the differences Δm_{ij}. These experimental data are from atmosphere neutrinos and solar neutrinos oscillations [100]. When the square difference in the atmosphere Δm_A is greater than zero, it is normal ordering, and it is inverted ordering when $\Delta m_A < 0$.

$$\begin{aligned} \text{normal}: m_2 &= \sqrt{m_1^2 + \Delta m^2_\odot},\ m_3 = \sqrt{m_1^2 + \Delta m^2_A} \\ \text{inverted}: m_2 &= \sqrt{m_3^2 + \Delta m^2_\odot + \Delta m^2_A},\ m_1 = \sqrt{m_3^2 + \Delta m^2_A} \end{aligned} \tag{4.56}$$

where $\Delta m_{\odot}^2$ and Δm_A^2 are the solar and atmosphere neutrino mass square difference, respectively. The hierarchy of the neutrino mass under different ordering are

$$\begin{aligned}\text{normal hierarchy}: m_3 \simeq \sqrt{\Delta m_A^2} \gg m_2 \simeq \sqrt{\Delta m_{\odot}^2} \gg m_1 \\ \text{inverted hierarchy}: m_2 \simeq m_1 \simeq \sqrt{\Delta m_A^2} \gg m_3 \\ \text{quasi-degeneracy}: m_0^2 \equiv m_1^2 \simeq m_2^2 \simeq m_3^2 \gg \Delta m_A^2\end{aligned} \tag{4.57}$$

Physicists usually draw the diagram of $\langle m_{ee} \rangle$ with respect to the lightest neutrino mass in different ordering and hierarchy to clearly observe the features of the effective mass. We show a brief plot in Figure 4.4. The figure has been plotted under the range of the mixing angle $\theta_{12}, \theta_{13}, \theta_{23}$, atmosphere/solar square mass differences and the CP violation phase ϕ [101, 102]. The review can be found here: [54].

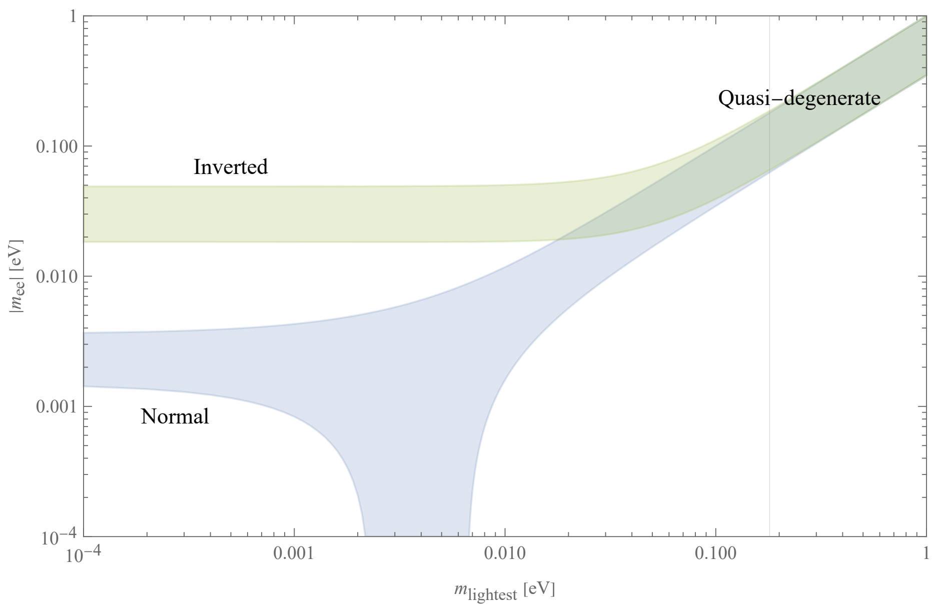


Figure 4.4: Normal: $|m_{ee}| - m_1$; Inverted: $|m_{ee}| - m_3$. The Quasi-degenerate part refers to the overlap of the two orderings. The shaded area range is defined by the minimum/maximum effective mass with varying Majorana phases ϕ_{12}, ϕ_{13}. The values of the parameters $c_{12}, c_{13}, s_{12}, s_{13}, \Delta m_{\odot}^2, \Delta m_A^2$ are taken from [55].

4.3 $0\nu\beta\beta$ in the Left-Right Symmetric Model

What if we use the mLRSM model to explain the neutrinoless double beta decay? The mLRSM permits the existence of the Majorana neutrinos which leads to the exchange of neutrino-antineutrino in the mediate propagation, i.e. it is reasonable to use mLRSM to calculate the cross section and the decay rate of neutrinoless double beta decay. The main work is to calculate the matrix element that sandwiches with the amplitude from the initial state to the final state. There are assumptions to make the matrix element physical. For instance, the s-matrix experiences a infinitesimal scattering time compared with the initial states coming from the infinity, and we use the interaction vacuum in the interaction picture. One can find these prerequisite knowledge in any of the quantum field theory text books in, e.g. [19]. In the effective theory, one can obtain the dimensionless coupling by using the negative mass dimension coefficients so as to make the Lagrangian perturbatively renormalizable [20].

We use Feynman diagrams to simplify the exploration of the amplitude. We have already seen from the experiments that the half-life limits of the neutrinoless double beta decays are really high. The possibility of finding these decays is somehow very small. So we only calculate the tree-level contributions in this thesis.

4.3.1 Tree-Level Contributions

Consider the neutrinoless double decay at the quark level $2d \to 2u + 2e^{-1}$. Therefore the tree level diagrams should have six legs with two in-going d quarks, two out-going u quarks, and two electrons. Due to the perturbation renormalizable theory, there should be four vertices. Here are the brief steps to calculate the Feynman rules and so as the amplitude: (1) For any given Lagrangian, one can use path integral to obtain the action and form the generating function. (2) Calculate the correlative functions by doing the functional derivative. For example, the two-point Green function is just the propagator. The rigorous process of looking for Feynman rules is somehow complex and not intuitive. However, we can observe the possible interactions directly from the Lagrangian (3.2). The kinetic terms have the form of a linear Green function and give the propagators under the choosing gauge. We could have fermion propagators, W/Z boson propagators and Higgs scalar propagators in our Lagrangian. The vertices, which are amputated (without the in-going and out-going momenta into the vertex), can be found in the fields interaction terms, e.g. in the lepton kinetic term we have $i\overline{L}\gamma^\mu(\vec{\sigma} \cdot \vec{W}_\mu)L$ for the lepton-W boson interactions. This shows a three-leg vertex with two leptons and one W boson. If we do the derivatives with respect to three fields, we obtain the vertex as well. In this matter of fact, we will have the interactions: lepton-W/Z boson, lepton-Higgs triplets, quark-boson, quark-Higgs, boson-boson self-interaction, and boson-Higgs. We can combine these external legs, propagators, and vertices to find all the permutations. There are books and papers for Feynman rules, for example, in the SM there are [30, 103, 104, 105], and in the mLRSM there is [67]. Details of the calculation of Feynman diagrams, especially the propagator of Majorana neutrinos, are in Appendix E.

The Feynman rule of the Majorana type propagator has a different form than the Dirac neutrino propagator. We show the result here, and one finds the derivation in Appendix E.

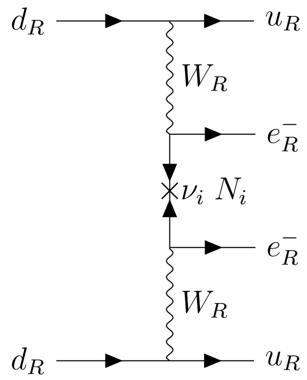
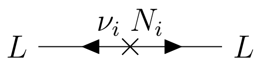
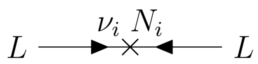


Figure 4.5: "Right-Right" case with two right-handed W bosons



$$P_L \frac{-i(\not{p} + m_{\nu,N})}{p^2 - m_{\nu,N}^2 + i\epsilon} P_L C \tag{4.58}$$



$$P_L \frac{-iC^{-1}(\not{p} + m_{\nu,N})}{p^2 - m_{\nu,N}^2 + i\epsilon} P_L \tag{4.59}$$

Let us draw Feynman diagrams. We have learned in the quantum field theory course that the Feynman diagrams do not carry physical meanings such as time flow, reaction flow, reaction procedure, etc. It is only a diagrammatic way to show the contributions. There are six tree-level diagrams contributions for different intermediate propagators. Four of them interact with two W bosons and contain neutrino propagators, see Figures 4.3,4.5, and 4.6,4.7. The other two include the Higgs boson as a propagator and W-Higgs interaction, see Figures 4.8 and 4.9.

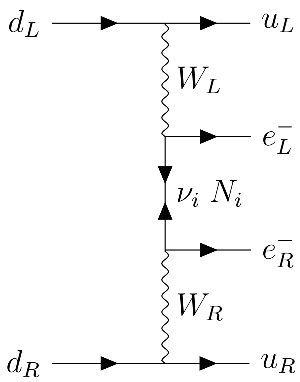


Figure 4.6: "Left-Right" case
λ-Contribution

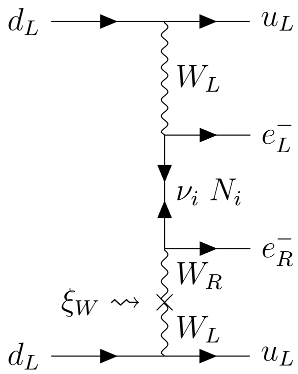


Figure 4.7: "Left-Right" case
η-Contribution

Let us determine the amplitude of each diagram. We have discussed the amplitude of the standard mass mechanism (4.43) in section 4.2. Now let us similarly write

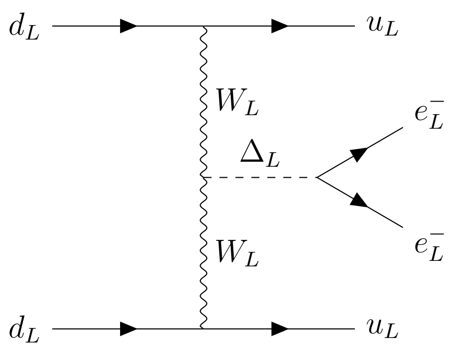


Figure 4.8: Left-handed charged Higgs scalar Δ_L

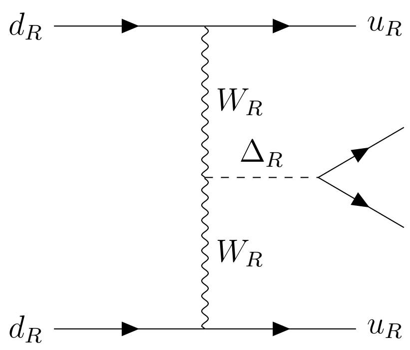


Figure 4.9: Right-handed charged Higgs scalar Δ_R

down the effective amplitude $\mathcal{A}$ of figures 4.5,4.6,4.7,4.8,4.9. We use the superscript to distinguish these equations. The complete discussion and approximation will be given when we calculate the differential decay rate in section 4.3.2.

Heavy neutrino propagator (RR case)

Following Equation (4.15) and (4.20), we could write down the effective amplitude of Figure 4.5.

$$\mathcal{A}_{eff}^{(RR)} = iG_F^2(\frac{m_{W_L}}{m_{W_R}})^4(\frac{U_{ei}^2}{m_{N_i}})V_{ud}^2 \cdot \overline{e^u}(\vec{p_{e1}})\gamma_\nu(1+\gamma^5)\gamma_\rho(e^u)^c(\vec{p_{e2}}) \cdot J_R^\nu J_R^\rho \tag{4.60}$$

where J_R^ν, J_R^ρ are the right-handed hadronic currents with $(1+\gamma^5)$ exchanged in Equation (4.19). In the heavy neutrino exchange case, the neutrino propagator can also be integrated out due to its large mass. The propagator therefore reduces to a constant, i.e. a vertex-like point. Since the Fermi constant G_F includes the left-handed W boson mass, we need a coefficient if we still want to use the Fermi constant when we integrate out the right-handed W boson. The chirality in each vertices are right-handed yet identical, which leaves the diagram unchanged after exchanging the momentum of the out-going electrons. This is the same argument with that in the LL case.

Light-heavy neutrino mixing (LR case)

There are two main processes: the λ-Contribution and the η-Contribution. The first aspect we pay attention to is the left-right chirality exchange in the LR process. The propagator includes light-heavy neutrino mixing. The mixing can be represented by the neutrinos mixing angle. There are different assumptions of the chirality exchange propagator. One of those is that the propagator contains both a light neutrino and a heavy neutrino [35]. The mechanism could be this: the "vacuum" generates the massive field through SSB. This nonlinear mass is described by the mass matrix. In another way of saying, Higgs boson interacts with the neutrino field to give it mass. Different chiral neutrinos are produced in the process and the left-right chirality exchange is one of the contributions. In this case, we could think about one Majorana neutrino $\overline{n}_L = (\overline{v}_L, \overline{(v_R)^c})$ with left-right mixing [36]. They mix through correlation angle transformed in mass eigenstates, see Equation (3.80). Under this assumption, we can write the neutrino propagator as

$$\begin{aligned} S_F^{LR} &= \langle 0| \nu_{eL}(x_1)\nu_{eR}(x_2) |0\rangle \\ &\approx \langle 0| \nu_{1L}(x_1) \sin\theta \nu_{1R}(x_2) |0\rangle \\ &\approx -\sin\theta \int \frac{d^4p}{(2\pi)^4} \mathrm{e}^{-ip\cdot(x_1-x_2)} P_L \frac{-i(\not{p}+m_\nu)}{p^2 - m_\nu^2 + i\epsilon} P_R C \end{aligned} \tag{4.61}$$

where we use the mixing,

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_{1L} \\ \nu_{2L} \end{pmatrix} = \begin{pmatrix} \nu_{eL} \\ \nu_{eR}^c \end{pmatrix} \tag{4.62}$$

and its conjugate,

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_{1R} \\ \nu_{2R} \end{pmatrix} = \begin{pmatrix} \nu_{eL}^c \\ \nu_{eR} \end{pmatrix} \tag{4.63}$$

ν_1, ν_2 are mass eigenstates with light and heavy mass, respectively, where we can see that the right-handed electron-type neutrino gives a coefficient $\sin\theta$ to the right-handed light neutrino part ν_{1R}, that leads a $\sin\theta$ in the propagator. For simplicity, we only consider one generation. In the general case, if we consider the conventional three generations, the transformation matrix is a 6×6 matrix. The transformation matrix can be written into a block matrix and the mixing angle is still between the correspondent generations. In (4.61), we also ignore the heavy neutrino contributions since it involves a heavy neutrino mass in the denominator, which leads to much smaller contributions compared to the light neutrino one. We can also simplify Equation (4.61) to the effective propagator, and write in the momentum space:

$$\tilde{S}_{ef}^{LR} \approx \sin\theta \frac{i\not{p}}{p^2} P_R C \tag{4.64}$$

where the in-going neutrinos are considered. The mass part in the nominator of the neutrino propagator vanishes while the momentum part remains.

$$P_L \frac{-i(\not{p}+m_{\nu_i})}{p^2 - m_{\nu_i}^2 + i\epsilon} P_R = \frac{-i\not{p}}{p^2 - m_{\nu_i}^2 + i\epsilon} P_R \tag{4.65}$$

When we consider about different channels (in $0\nu\beta\beta$ in our thesis, there are s-channel and u-channel), we need to exchange the out-going momentum of out-going electrons ($p_1 \longleftrightarrow p_2$). In the propagator (4.64), this keeps the 0-component and adds a minus

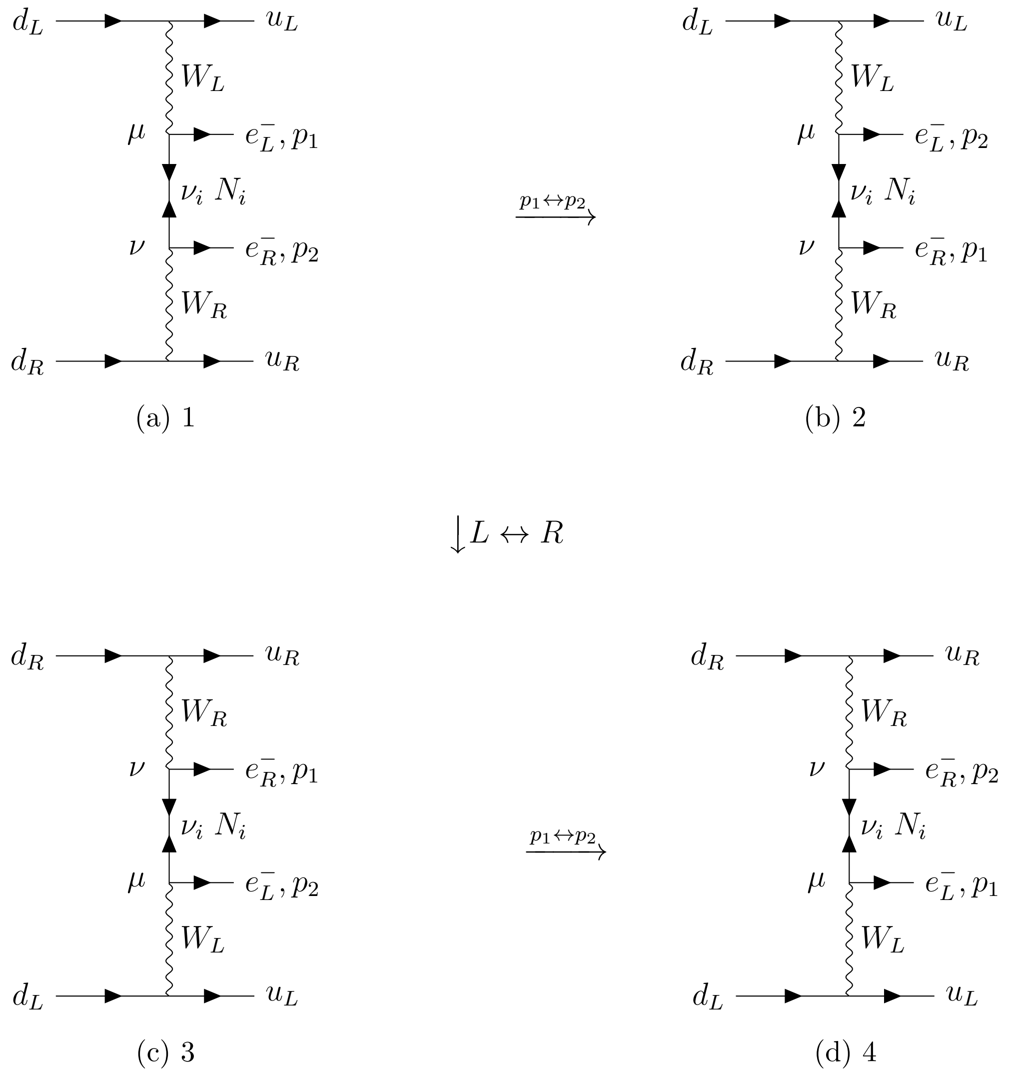


Figure 4.10: 4 λ-Contributions of LR case

sign to the spatial part.

$$\frac{p^0\gamma_0 - \vec{p}\cdot\vec{\gamma}}{p^2} - \frac{p^0\gamma_0 + \vec{p}\cdot\vec{\gamma}}{p^2} \tag{4.66}$$

There are more contributions as well in LR case: the chirality at each vertices in Figure 4.6 can be L-R or R-L from the upper to the lower. All the contributions of LR case are in Figure 4.10. From the Figures 4.10a to 4.10d, we write down the amplitudes of the correspondent leptonic parts with hadronic currents.

$$1.\ \overline{u}(p_1)(1+\gamma_5)\gamma_\mu \not{p}\gamma_\nu u^c(p_2)J_L^\mu(x_1)J_R^\nu(x_2) \tag{4.67}$$

$$2.\ [\overline{u}(p_2)(1+\gamma_5)\gamma_\mu(\gamma_0 p^0+\vec{\gamma}\cdot\vec{p})\gamma_\nu u^c(p_1)]^T J_L^\mu(x_2)J_R^\nu(x_1) \tag{4.68}$$

$$3.\ \overline{u}(p_1)(1-\gamma_5)\gamma_\nu \not{p}\gamma_\mu u^c(p_2)J_R^\nu(x_1)J_L^\mu(x_2) \tag{4.69}$$

$$4.\ [\overline{u}(p_2)(1-\gamma_5)\gamma_\nu(\gamma_0 p^0+\vec{\gamma}\cdot\vec{p})\gamma_\mu u^c(p_1)]^T J_R^\nu(x_2)J_L^\mu(x_1) \tag{4.70}$$

where the effective amplitude includes the same coefficient $iG_F^2\frac{m_{W_L}^2}{m_{W_R}^2}V_{ud}^2\frac{1}{p^2}$. We also change the Lorentz indices of gamma matrices from contravariant to covariant so that the indices are placed in the same subscript. This change is mathematically safe since the amplitude or decay rates are scalars, i.e. all the indices will be contracted.

Next, we consider the η-Contribution. The η-diagrams also contain the exchange of the different chiral electrons and the chirality exchange of the lepton-boson vertices. Therefore we should also have 4 diagrams similar to Figure 4.10 and the leptonic part of the two contributions diagram should be identical. The differences come from: (1) the NME and (2) an additional $W_L - W_R$ mixing angle ξ_W from Equation (3.41). The η-diagrams are in Figure 4.11. We can easily write down the electron wave function and the hadronic current parts following (4.67) to (4.70):

$$1.\ \overline{u}(p_1)(1+\gamma_5)\gamma_\mu \not{p}\gamma_\nu u^c(p_2)J_L^\mu(x_1)J_L^\nu(x_2) \tag{4.71}$$

$$2.\ [\overline{u}(p_2)(1+\gamma_5)\gamma_\mu(\gamma_0 p^0+\vec{\gamma}\cdot\vec{p})\gamma_\nu u^c(p_1)]^T J_L^\mu(x_2)J_L^\nu(x_1) \tag{4.72}$$

$$3.\ \overline{u}(p_1)(1-\gamma_5)\gamma_\nu \not{p}\gamma_\mu u^c(p_2)J_L^\nu(x_1)J_L^\mu(x_2) \tag{4.73}$$

$$4.\ [\overline{u}(p_2)(1-\gamma_5)\gamma_\nu(\gamma_0 p^0+\vec{\gamma}\cdot\vec{p})\gamma_\mu u^c(p_1)]^T J_L^\nu(x_2)J_L^\mu(x_1) \tag{4.74}$$

where the coefficient $iG_F^2V_{ud}^2\sin\xi_W\frac{1}{p^2}$ has been taken away just for simplicity.

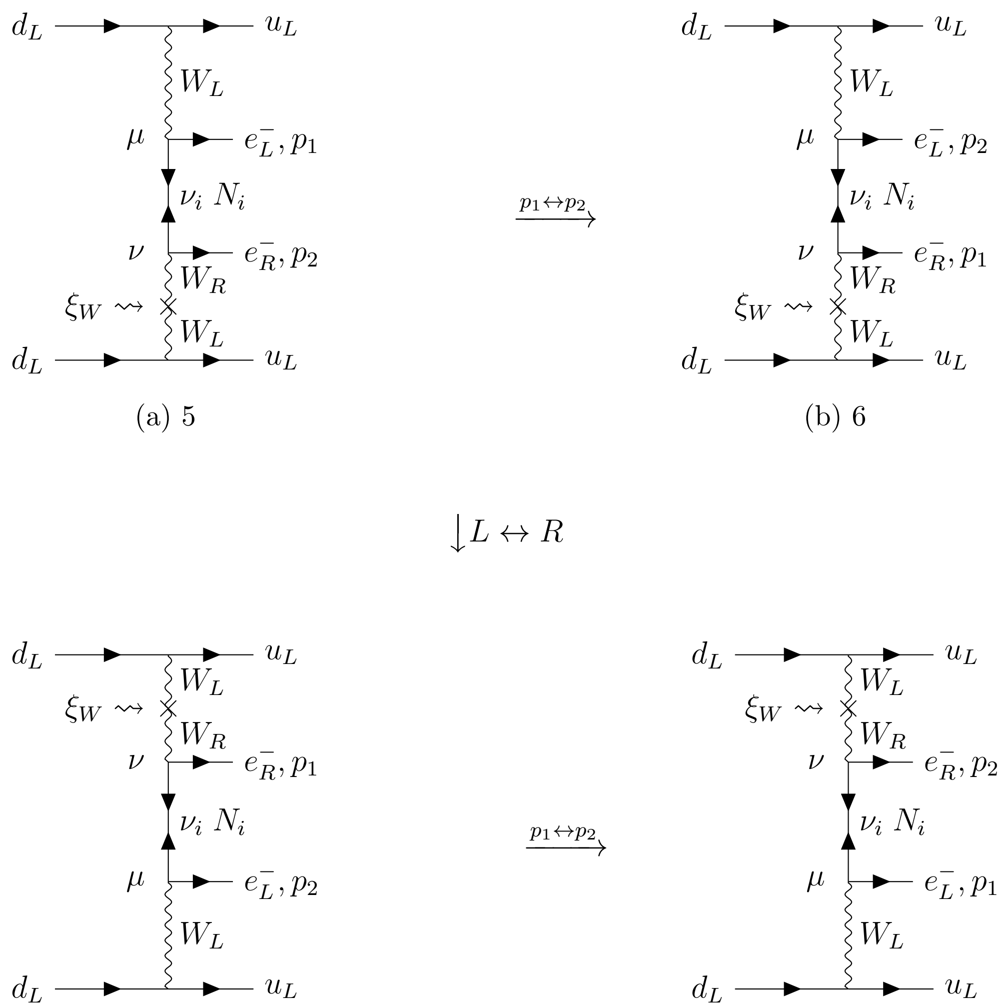
Charged Higgs scalar intermediate interaction

In Appendix E we give the Feynman rules for the W-Higgs vertex and the lepton-Higgs vertex. We can assume that the Higgs scalars do not propagate. The scalar type propagator of Higgs bosons "squeeze" to a point and left the constant which is proportional to reciprocal of the charged Higgs field. We have obtained the two charged mass in Equation (3.31). The amplitudes are

$$\mathcal{A}_{eff}^{\Delta_L} = 8iG_F^2V_{ud}^2\frac{1}{\nu_R^2}\frac{\nu_L}{\nu_R}\overline{u^s}(p_1)Y_{ee}P_L u^{s'}(p_2)J_{\mu L}(x_1)J_L^\mu(x_2) \tag{4.75}$$

$$\mathcal{A}_{eff}^{\Delta_R} = 8iG_F^2V_{ud}^2\frac{1}{\nu_R^2}\overline{u^s}(p_1)Y_{ee}P_R u^{s'}(p_2)J_{\mu R}(x_1)J_R^\mu(x_2) \tag{4.76}$$

where the VEVs in the nominator come from the W boson-Higgs vertex and Y_{ee} is the first component of the diagonalized Yukawa matrix (E.5) and (E.4). However, it is not tough to write down the effective amplitude from Feynman rules for all



(a) 5 (b) 6

(c) 7 (d) 8

Figure 4.11: 4 η-Contributions of LR case

the diagrams. In this section we only give a quick glance at how we can obtain the effective amplitude. In the next section, we will calculate the scattering matrix and the differential decay rate step by step.

Combining the diagrams and amplitude expressions, we can match with the 4-fermions effective Lagrangian. The couplings would include the mechanisms. The effective Lagrangian can be explicitly calculated from the path integral of the effective action. However, one can also simply write down the effective Lagrangian from each diagram, i.e. the 6 dimensional effective Lagrangian should be proportional to $\overline{u}_{L,R}\gamma_\mu d_{L,R}\overline{l}_{L,R}\gamma^\mu l_{L,R}$ and the 9 dimensional effective Lagrangian should be proportional to three currents $\overline{u}_{L,R}\gamma_\mu d_{L,R}\overline{u}_{L,R}\gamma_\mu d_{L,R}\overline{e}_{L,R}\gamma^\mu e_{L,R}$, where l can be the charged lepton field or the neutrino field. There is only the first order of the hadronic currents that the pure $V \pm A$ is taken into account. We can define the effective Lagrangian as

$$\begin{aligned}\mathcal{L}_{eff} =& \frac{G_F}{\sqrt{2}}(\epsilon_{LL} j_{L\mu} J_L^\mu + \epsilon_{RR} j_{R\mu} J_R^\mu + \lambda j_{R\mu} J_R^\mu + \eta j_{R\mu} J_L^\mu) \\ &+ \frac{G_F^2}{\sqrt{2} m_p}(\epsilon_{\Delta_L} j_L J_{L\mu} J_L^\mu + \epsilon_{\Delta_R} j_R J_{R\mu} J_R^\mu)\end{aligned} \tag{4.77}$$

where m_p is the proton mass. We have used the Fierz identity in forming the Lagrangian, more details see Appendix D. The hadronic currents J are defined in Equations (4.19) and (4.60), and the leptonic currents j are defined as

$$\begin{aligned} \bar{e}\gamma^\mu(1-\gamma^5)\nu \equiv j_L^\mu \qquad & \bar{e}\gamma^\mu(1+\gamma^5)\nu \equiv j_R^\mu \\ \bar{e}(1-\gamma^5)e^c \equiv j_L \qquad & \bar{e}(1+\gamma^5)e^c \equiv j_R \end{aligned} \tag{4.78}$$

The coefficients λ and η are couplings of λ-contribution and η-contribution respectively. They are not the same couplings as those in $2\nu\beta\beta$ in Equation (4.8). The couplings in our derivations are

$$\begin{aligned} \epsilon_{LL} &= \sum_i V_{ud} U_{ei} \frac{m_{\beta\beta}}{m_e} & \epsilon_{RR} &= \sum_i V_{ud} U_{ei} \frac{m_p}{|m_{N_i}|} \frac{m_{W_L}^4}{m_{W_R}^4} \\ \lambda &= \sum_i V_{ud} U_{ei} \frac{m_{W_L}^2}{m_{W_R}^2} \sin\theta_\nu & \eta &= \sum_i V_{ud} U_{ei} \sin\xi_W \\ \epsilon_{\Delta_L} &= \sum_i V_{ud} \frac{Y_{ee} m_p}{\nu_R^2} \frac{\nu_L}{\nu_R} & \epsilon_{\Delta_R} &= \sum_i V_{ud} \frac{Y_{ee} m_p}{\nu_R^2} \frac{m_{W_L}^4}{m_{W_R}^4} \end{aligned} \tag{4.79}$$

where Y_{ee} is the electron component of the diagonalized Yukawa matrix in section 3.5, see Equation (3.73). One can also find some hints in other papers, e.g. [12]. These couplings can be constrained by the experimental half-life, see section 4.4.

4.3.2 Derivation of the Decay Rates

The Standard mass mechanism (LL case)

We have shown the derivation of standard mass mechanism decay rate in section 4.2.

Heavy neutrino propagator (RR case)

Let us derive the differential decay rate in the RR case that includes a heavy neutrino as the intermediate propagator. The heavy neutrino only appears in the BSM when the right correspondent symmetries or hyper symmetries are introduced. The heavy mass we utilize here is proportional to the right-handed Higgs VEVs (3.86) from the seesaw mechanism. This offers a constraint on the heavy neutrino mass scale. Moreover, the argument of the heavy neutrino mass in section (3.5) manifests $m_{N_i} \gtrsim 300$GeV, which is much greater than the neutrino momentum. The mass part dominates in the denominator of the neutrino propagator (E.17) with heavy mass, $p^0 = \sqrt{\vec{p}^2 + m_{N_i}^2} \approx m_{N_i}$. Effectively, each right-handed propagator provides a coefficient $(\frac{m_{W_L}}{m_{W_R}})^2$ since the Fermi constant is with respect to m_{W_L}. Due to the parity invariant in the mLRSM, the couplings g_L and g_R are considered equal. In the RR case, Figure 4.5, the analysis of chirality projection operator is equivalent to the LL case. It is not difficult to write down the matrix element of the RR case in perturbation theory. Compared to Equation (4.29), the RR matrix element is expressed as

$$\begin{aligned}\langle f| \mathcal{A}_{eff}^{(RR)} |i\rangle =& iG_F^2(\frac{m_{W_L}}{m_{W_R}})^4 V_{ud}^2 \cdot N_{p_1} N_{p_2} \overline{u^s}(p_1)\gamma_\nu(1+\gamma^5)\gamma_\rho v^{s'}(p_2) \\ &\times \int d^3x_1 d^3x_2 \mathrm{e}^{-i\vec{p}_1\cdot\vec{x}_1 - i\vec{p}_2\cdot\vec{x}_2} \cdot \frac{\sum_i U_{ei}}{m_{N_i}} \frac{1}{(2\pi)^3} \int d^3p \mathrm{e}^{i\vec{p}\cdot(\vec{x_1}-\vec{x_2})} \\ &\times \big[\sum_n \langle f| J_R^\nu(x_1) |N_n\rangle \langle N_n| J_R^\rho(x_2) |i\rangle + \langle f| J_R^\nu(x_2) |N_n\rangle \langle N_n| J_R^\rho(x_1) |i\rangle \big] \\ &\times 2\pi\delta(E_f + p_1^0 + p_2^0 - E_i) - (p_1 \longleftrightarrow p_2) \end{aligned} \tag{4.80}$$

Due to the large mass, the propagation will not last too long. The long wave approximation thereby is still available. The denominator for the nuclear part reduces to m_{N_i} due to its large mass and is combined with the neutrino part. Additionally, the two hadronic current correlation remains unchanged for the sign.

$$\begin{aligned}&\vec{J}_{\mu R}(\vec{x}_1)\vec{J}_R^\mu(\vec{x}_2) \\ &= \sum_{n,m} \delta(\vec{x}_1 - \vec{r}_m)\delta(\vec{x}_2 - \vec{r}_n)\tau_+^n\tau_+^m[g_V(q^2)g^{\mu 0} - g_A(q^2)\sigma_i^n g^{\mu i}][g_V(q^2)g_{\mu 0} - g_A(q^2)\sigma_i^m g_{\mu i}] \\ &= \sum_{n,m} \delta(\vec{x}_1 - \vec{r}_n)\delta(\vec{x}_2 - \vec{r}_m)\tau_+^n\tau_+^m[g_V^2 - \vec{\sigma}^n\vec{\sigma}^m g_A^2] \end{aligned} \tag{4.81}$$

The exchange of the momentum of the out-going electrons in RR case also produce a factor 2. Next we should redefine the neutrino potential function. Similarly do the integral as in (4.36).

$$\frac{1}{(2\pi)^3}\int d^3p \mathrm{e}^{i\vec{p}\cdot(\vec{x_1}-\vec{x_2})} = \frac{m_e m_p}{4\pi R} H^{(RR)}(r_{nm}, \overline{E_n}) \tag{4.82}$$

where the definitions of parameters are same as in the LL case (4.36). The electron mass m_e and proton mass m_p are introduced to keep the NME dimensionless, see RR case NME (4.85) [12, 106]. The neutrino potential is

$$H^{(RR)}(r_{nm}, \overline{E_n}) = \frac{1}{m_e m_p}\frac{2R}{\pi r_{nm}}\int_0^\infty \frac{p^2 \sin p r_{nm}}{p} dp \tag{4.83}$$

We can now insert Equation (4.82) and (4.83) into the Equation (4.80). It becomes

$$\begin{aligned}\langle f| \mathcal{A}_{eff}^{(RR)} |i\rangle =& iG_F^2(\frac{m_{W_L}}{m_{W_R}})^4 V_{ud}^2 \cdot N_{p_1} N_{p_2} \overline{u^s}(p_1)(1-\gamma^5)v^{s'}(p_2) \\ & \frac{\sum_i U_{ei}^2}{m_{N_i}} \frac{m_e m_p}{2R} \cdot \sum_{n,m} \langle f| H^{(RR)}(r_{nm}, \overline{E_n}) \tau_+^n \tau_+^m [\vec{\sigma}^n \vec{\sigma}^n g_A^2 - g_V^2] |i\rangle \\ & \delta(E_f + p_1^0 + p_2^0 - E_i)\end{aligned} \tag{4.84}$$

The definition of the NME is unlike the standard mechanism one. This is the consequence of the large mass scale of the heavy neutrino comparing to the decay energy.

$$\begin{aligned} M_{0\nu}^{(RR)} &= M_{0v,GT}^{(RR)} - \frac{g_V^2}{g_A^2} M_{0v,F}^{(RR)} \\ M_{0v,GT}^{(RR)} &\equiv \sum_{n,m} \langle N_f| H^{(RR)}(r_{nm}, \overline{E_n}) \tau_+^n \tau_+^m \vec{\sigma}^n \cdot \vec{\sigma}^m |N_i\rangle \\ M_{0v,F}^{(RR)} &\equiv \sum_{n,m} \langle N_f| H^{(RR)}(r_{nm}, \overline{E_n}) \tau_+^n \tau_+^m |N_i\rangle \end{aligned} \tag{4.85}$$

Put Equation (4.85) into (4.84), use trace technology and calculate the absolute square. Although the leptonic part in the RR matrix element has $1-\gamma^5$ in difference, the absolute square of this part is equivalent. The absolute square is

$$\begin{aligned}\left|\langle f| \mathcal{A}_{eff}^{(RR)} |i\rangle\right|^2 =& (\frac{\sum_i U_{ei}^2}{m_{N_i}})^2 (\frac{m_{W_L}}{m_{W_R}})^8 m_e^2 m_p^2 \frac{G_F^4 V_{ud}^4 g_A^4}{2(2\pi)^6 E_1 E_2 R^2} (E_1 E_2 - p_1 p_2 \cos\xi) \\ & \times \left|M_{0\nu}^{(RR)}\right|^2 \delta(E_f + p_1^0 + p_2^0 - E_i)\end{aligned} \tag{4.86}$$

define $\frac{1}{\langle m_N \rangle} \equiv \frac{\sum_i U_{ei}^2}{m_{N_i}}$. The differential rate is

$$\begin{aligned} d\Gamma_{(RR)}^{0\nu} =& \left|\frac{1}{\langle m_N \rangle}\right|^2 \left|M_{0\nu}^{(RR)}\right|^2 m_e^2 m_p^2 (\frac{m_{W_L}}{m_{W_R}})^8 \frac{G_F^4 V_{ud}^4 g_A^4}{(2\pi)^5 R^2} (E_1 E_2 - p_1 p_2 \cos\xi) \\ & \times F(Z+2, E_1) F(Z+2, E_2) p_1 p_2 \sin\xi d\xi dE_2 \end{aligned} \tag{4.87}$$

where $E_1 = p_1^0, E_2 = p_2^0$ are electron energies.

Light-heavy neutrino mixing (LR case)

λ-Contribution. We have showed the four contributions in Figure (4.10). The amplitude of the lepton part with hadronic currents is in Equation (4.70). Let us first simplify the amplitude. The exchange of the out-going fermions provides an additional minus sign, which is the same mechanism of changing the field position in wick theorem [19]. Therefore, we combine the four equations.

$$\begin{aligned}
&1-2+3-4\\
=&+\overline{u}(p_1)(1+\gamma_5)\gamma_\mu \not{p}\gamma_\nu u^c(p_2)J_L^\mu(x_1)J_R^\nu(x_2)\\
&-\overline{u}(p_1)(1-\gamma_5)\gamma_\nu(\gamma_0 p^0+\vec{\gamma}\cdot\vec{p})\gamma_\mu u^c(p_2)J_L^\mu(x_2)J_R^\nu(x_1)\\
&+\overline{u}(p_1)(1-\gamma_5)\gamma_\nu \not{p}\gamma_\mu u^c(p_2)J_R^\nu(x_1)J_L^\mu(x_2)\\
&-\overline{u}(p_1)(1+\gamma_5)\gamma_\mu(\gamma_0 p^0+\vec{\gamma}\cdot\vec{p})\gamma_\nu u^c(p_2)J_R^\nu(x_2)J_L^\mu(x_1)
\end{aligned} \tag{4.88}$$

The momentum exchange from 1 to 2 also defines the time ordering in the hadronic current. Combining the first line and the last line, We will have the hadronic current (1-4) with the potential integral as

$$\begin{aligned}
\int d^3x_1 d^3x_2 \mathrm{e}^{-i(\vec{p_1}\cdot\vec{x_1}+\vec{p_2}\cdot\vec{x_2})}\frac{1}{2(2\pi)^3}\int d^3p\Big[&\frac{\not{p}\mathrm{e}^{i\vec{p}\cdot(\vec{x}_1-\vec{x}_2)}}{p^0}\frac{\langle f|\, J_L^\mu(\vec{x}_1)J_R^\nu(\vec{x}_2)\,|i\rangle}{\vec{p}+p_2^0+\overline{E_n}-M_i+i\epsilon}\\
&-\frac{(\gamma_0 p^0+\vec{\gamma}\cdot\vec{p})\mathrm{e}^{-i\vec{p}\cdot(\vec{x}_1-\vec{x}_2)}}{p^0}\frac{\langle f|\, J_R^\nu(\vec{x}_2)J_L^\mu(\vec{x}_1)\,|i\rangle}{\vec{p}+p_1^0+\overline{E_n}-M_i+i\epsilon}\Big]
\end{aligned} \tag{4.89}$$

We have use the residue theorem 4.22 and thus only the on-shell condition has been taken into account. In the amplitude $\langle f|\,\mathcal{A}_{eff}^{(LR-\lambda)}\,|i\rangle$, we need to add the coefficient, the electron wave function, the chirality operator, and the neutrino mixing angle. The amplitude is given explicitly in the subsequent calculation. Let us now focus on the integral. If only the S-wave of $0^+\to 0^+$ is considered, the approximation $\mathrm{e}^{-i(\vec{p_1}\cdot\vec{x_1}+\vec{p_2}\cdot\vec{x_2})}\approx 1$ can be used here as well. However, it is necessary to consider the first order expansion of the electron wave function: $\mathrm{e}^{-i(\vec{p_1}\cdot\vec{x_1}+\vec{p_2}\cdot\vec{x_2})}\approx 1-i(\vec{p_1}\cdot\vec{x_1}+\vec{p_2}\cdot\vec{x_2})$ instead of only the zeroth order, if we think about the relativistic correction of the hadronic current [107]. We will only consider the zeroth order of the expansion and the p^0 component of the neutrino momentum for the pure $0^+\to 0^+$ transition in this thesis. The spatial component $\vec{p}$ provides an odd parity while the two electrons provide an even parity, which the final states are indeed not 0^+ states. Let us still consider the non-relativistic impulse approximation of the hadronic current, see Equation (4.30). The integral becomes

$$\begin{aligned}
&\frac{1}{2(2\pi)^3}\int d^3p\Big[\frac{\gamma_0\mathrm{e}^{i\vec{p}\cdot(\vec{r}_n-\vec{x}_m)}}{\vec{p}+p_2^0+\overline{E_n}-M_i}-\frac{\gamma_0\mathrm{e}^{i\vec{p}\cdot(\vec{r}_n-\vec{r}_m)}}{\vec{p}+p_1^0+\overline{E_n}-M_i}\Big]\\
&\cdot\langle f|\sum_{n,m}\tau_+^n\tau_+^m[g_V(q^2)g^{\mu 0}+g_A(q^2)\sigma_i^n g^{\mu i}][g_V(q^2)g^{\nu 0}-g_A(q^2)\sigma_i^m g^{\nu i}]\,|i\rangle
\end{aligned} \tag{4.90}$$

We can further simplify the integration part to

$$\frac{1}{2(2\pi)^3}\int d^3p\gamma_0\mathrm{e}^{i\vec{p}\cdot(\vec{r}_n-\vec{r}_m)}\frac{p_1^0-p_2^0}{(\vec{p}+A_1)(\vec{p}+A_2)} \tag{4.91}$$

times the hadronic current part, where $A_{1,2} \equiv p^0_{1,2} + \overline{E_n} - M_i$. In the intermediate process with $\overline{E_n}$, the momentum of the neutrino is usually > 20MeV while $A_{1,2}$ are usually only a few MeV [107]. Therefore, the integration (4.91) has the following relation with the neutrino potential (4.37) in the LL case,

$$\frac{1}{2(2\pi)^3}\int d^3p\gamma_0 \mathrm{e}^{i\vec{p}\cdot(\vec{r}_n-\vec{x}_m)}\frac{p^0_1-p^0_2}{(\vec{p}+A_1)(\vec{p}+A_2)} \approx \gamma_0(p^0_1-p^0_2)\frac{1}{2(2\pi)^3}\int d^3p\frac{\mathrm{e}^{i\vec{p}\cdot(\vec{r}_n-\vec{x}_m)}}{\vec{p}(\vec{p}+A)}$$
$$=\gamma_0(p^0_1-p^0_2)\frac{1}{8\pi R}H(r_{nm},\overline{E_n}) \tag{4.92}$$

where $A \equiv p + \overline{E_n} - \frac{M_i+M_f}{2}$. $H(r_{nm},\overline{E_n})$ is the neutrino potential function in LL case, see Equation (4.38). Inserting the integral (4.91) into (4.90) and combining the chirality in Equation (4.88), we have

$$\frac{1}{8\pi R}[(1+\gamma_5)p^0_1-(1+\gamma_5)p^0_2]\times H(r_{nm},\overline{E_n})$$
$$\times \langle f|\sum_{n,m}\tau^n_+\tau^m_+[g_V(q^2)g^{\mu 0}+g_A(q^2)\sigma^n_i g^{\mu i}][g_V(q^2)g^{\nu 0}-g_A(q^2)\sigma^m_i g^{\nu i}]\,|i\rangle \tag{4.93}$$

where $A_{1,2} \equiv -p^0_{1,2} - \overline{E_n} + M_i$. We use the same calculation procedure for terms (3-2) in (4.88). Considering the chirality operator and the Lorentz indices in the hadronic current, we will obtain

$$\frac{1}{8\pi R}[(1-\gamma_5)p^0_1-(1-\gamma_5)p^0_2]\times H(r_{nm},\overline{E_n})$$
$$\times \langle f|\sum_{n,m}\tau^n_+\tau^m_+[g_V(q^2)g^{\nu 0}-g_A(q^2)\sigma^n_i g^{\nu i}][g_V(q^2)g^{\mu 0}+g_A(q^2)\sigma^m_i g^{\mu i}]\,|i\rangle \tag{4.94}$$

The contraction of the Lorentz indices between the gamma matrices and the currents is derived using the anticommutation of the gamma matrices from the Clifford algebra (4.32) and (F.3). Collect all the gamma matrices and use one of the orders $\mu\nu$ as an example,

$$\begin{aligned}\gamma_\mu\gamma_0\gamma_\nu J^\mu J^\nu =&\gamma_0\gamma_0\gamma_\nu J^0 J^\nu - \gamma_k\gamma_0\gamma_\nu J^k J^\nu\\ =&\gamma_0\gamma_0\gamma_\nu J^0 J^\nu + \gamma_0\gamma_k\gamma_\nu J^k J^\nu\\ =&\gamma_0 J^0 J_0 - \gamma_0\delta^j_k J^k J_j\\ =&\gamma_0\,\langle f|\sum_{n,m}\tau^n_+\tau^m_+[g^2_V(q^2)-g^2_A(q^2)(\vec{\sigma}^m\cdot\vec{\sigma}^n)]\,|i\rangle\end{aligned} \tag{4.95}$$

The subscript k refers to the spacial part. In the third line, we also change the metric $g_{kj} = dim\{-1,-1,-1\}$ to a delta symbol for the Euclidean summation. The $\frac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$ part does not give a contribution since the hadronic currents are commute in the summation. Then we can define the NME with the same form in the standard mechanism, see (4.40).

$$M_{0\nu}^{(LR)} = M_{0\nu}^{(LL)} \tag{4.96}$$

We will keep the indices (LL) and (LR) to make the results clearly match the processes. Combining Equations (4.93) and (4.94) with the Lorentz indices and the NMEs, we have

$$\frac{1}{4\pi R}\gamma_0(p_1^0 - p_2^0)g_A^2 M_{0v}^{(LL)} \tag{4.97}$$

Now we retrieve the coefficient and the wave function to obtain the full expression of the matrix element.

$$\begin{aligned}\langle f|\,\mathcal{A}_{eff}^{(LR-\lambda)}\,|i\rangle =&G_F^2(\frac{m_{W_L}^2}{m_{W_R}^2})V_{ud}^2 g_A^2 \sin\theta_\nu \cdot N_{p_1}N_{p_2}\frac{1}{4\pi R}(p_1^0 - p_2^0)\overline{u_s}(p_1)\gamma_0 u_{s'}^c(p_2)\\ &\times M_{0\nu}^{(LR)}\cdot 2\pi\delta(E_f + p_1^0 + p_2^0 - E_i)\end{aligned} \tag{4.98}$$

The absolute square of (4.98) after using trace technology is

$$\begin{aligned}\left|\langle f|\,\mathcal{A}_{eff}^{(LR-\lambda)}\,|i\rangle\right|^2 =&(\frac{m_{W_L}}{m_{W_R}})^4 \sin^2\theta_\nu \frac{G_F^4 V_{ud}^4 g_A^4}{4(2\pi)^6 p_1^0 p_2^0 R^2}(p_1^0 - p_2^0)^2(p_1^0 p_2^0 + p_1 p_2\cos\xi - m_e^2)\\ &\times \left|M_{0\nu}^{(LR)}\right|^2 \delta(E_f + p_1^0 + p_2^0 - E_i)\end{aligned} \tag{4.99}$$

where the trace technology of the leptonic part is

$$\frac{1}{4}\mathrm{Tr}[(\not{p}_1 + m_e)\gamma_0(\not{p}_2 - m_e)\gamma_0] = p_1^0 p_2^0 + p_1 p_2 \cos\xi - m_e^2 \tag{4.100}$$

where the coefficient $\frac{1}{4}$ is to remove the overcounting of spin permutations. The differential decay rate is

$$\begin{aligned}d\Gamma_{(LR-\lambda)}^{0\nu} =&\sin^2\theta_\nu(\frac{m_{W_L}}{m_{W_R}})^4\left|M_{0\nu}^{(LR)}\right|^2\frac{G_F^4 V_{ud}^4 g_A^4}{2(2\pi)^5 R^2}(E_1 - E_2)^2(E_1E_2 - m_e^2 + p_1 p_2\cos\xi)\\ &\times F(Z+2, E_1)F(Z+2, E_2)p_1 p_2 \sin\xi d\xi dE_2\end{aligned} \tag{4.101}$$

where we rewrite the electron energies $p_1^0 = E_1, p_2^0 = E_2$.

η-Contribution. From Equation (4.70) and (4.74) we can clearly find out that the neutrino potential and the trace of the electron spin states of both contributions are identical. However, there would be a plus sign in between the time ordering of the hadronic current, since the hadronic part satisfies $J_L^\mu(x_1)J_L^\nu(x_2) \equiv J_L^\mu(x_2)J_L^\nu(x_1)$ when contracted with $g_{\mu\nu}$. Thus, we will have the same NMEs as those in the standard mechanism LL case (4.40). The amplitude in this way is

$$\begin{aligned}\langle f|\,\mathcal{A}_{eff}^{(LR-\eta)}\,|i\rangle =& G_F^2 V_{ud}^2 g_A^2 \sin\theta_\nu \sin\xi_W \cdot N_{p_1} N_{p_2} \frac{1}{4\pi R}(p_1^0 + p_2^0)\overline{u_s}(p_1)\gamma_0 u_{s'}^c(p_2) \\ & \times M_{0\nu}^{(LR-\eta)} \cdot 2\pi\delta(E_f + p_1^0 + p_2^0 - E_i)\end{aligned} \tag{4.102}$$

where the NME is defined as

$$M_{0\nu}^{(LR-\eta)} = M_{0\nu,GT}^{(LR)} + \frac{g_V^2}{g_A^2} M_{0\nu,F}^{(LR)} \tag{4.103}$$

The GT and F parts of the NME have the same definition as the λ-contribution. The differential decay rate is

$$\begin{aligned}d\Gamma_{(LR-\eta)}^{0\nu} =& \sin^2\theta_\nu \sin^2\xi_W \left| M_{0\nu}^{(LR-\eta)} \right|^2 \frac{G_F^4 V_{ud}^4 g_A^4}{2(2\pi)^5 R^2}(E_1 + E_2)^2 (E_1 E_2 - m_e^2 + p_1 p_2 \cos\xi) \\ & \times F(Z+2, E_1) F(Z+2, E_2) p_1 p_2 \sin\xi d\xi dE_2\end{aligned} \tag{4.104}$$

where we again rewrite the electron energies $E_{1,2} \equiv p_{1,2}^0$.

Charged Higgs scalar intermediate interaction

When engage with Higgs scalars, we have two contributions: Figures 4.8 and 4.9. The transition matrix elements combined with the quation (4.76) and the approximations becomes

$$\begin{aligned}\langle f|\,\mathcal{A}_{eff}^{\Delta_L}\,|i\rangle =& i G_F^2 V_{ud}^2 g_A^2 N_{p_1} N_{p_2} Y_{ee} \frac{1}{\nu_R^2} \frac{\nu_L}{\nu_R} \frac{1}{R} m_e m_p \overline{u^s}(p_1)(1-\gamma^5) u^{s'}(p_2) \\ & \times M_{0\nu}^{\Delta_L} \delta(E_f + p_1^0 + p_2^0 - E_i)\end{aligned} \tag{4.105}$$

$$\begin{aligned}\langle f|\,\mathcal{A}_{eff}^{\Delta_R}\,|i\rangle =& i G_F^2 V_{ud}^2 g_A^2 N_{p_1} N_{p_2} Y_{ee} (\frac{m_{W_L}}{m_{W_R}})^4 \frac{1}{\nu_R^2} \frac{1}{R} m_e m_p \overline{u^s}(p_1)(1+\gamma^5) u^{s'}(p_2) \\ & \times M_{0\nu}^{\Delta_R} \delta(E_f + p_1^0 + p_2^0 - E_i)\end{aligned} \tag{4.106}$$

where the NME of process $0^+ \to 0^+$ has the similar definition as RR case. This is obvious since we consider the non-propagating heavy particle mass scale in effective theory. The differences come from the coupling at the vertices and the mass scale of the W boson propagator. The left- and right-handed Higgs scalar cases are identical in the NME as follows.

$$\begin{aligned} M_{0\nu}^{(\Delta_{L/R})} &= M_{0v,GT}^{(\Delta_{L/R})} - \frac{g_V^2}{g_A^2} M_{0v,F}^{(\Delta_{L/R})} \\ M_{0v,GT}^{(\Delta_{L/R})} &= M_{0v,GT}^{(RR)} \qquad M_{0v,F}^{(\Delta_{L/R})} = M_{0v,F}^{(RR)} \end{aligned} \tag{4.107}$$

where the minus sign in the first line is for the left current NME and the plus sign is for the right counterpart. Since the Higgs-lepton vertex coupling provides a neutrino mass, we only need one mass unit to compensate the unit in phase space factor, where we set the dimension of the phase space factor to $[t^{-1}]$. The absolute squares with trace technology are

$$\begin{aligned}\left|\langle f|\,\mathcal{A}_{eff}^{\Delta_L}\,|i\rangle\right|^2 =&(Y_{ee})^2 m_e^2 m_p^2 \frac{1}{\nu_R^4}\frac{\nu_L^2}{\nu_R^2}\frac{2g_A^4 G_F^4 V_{ud}^4}{(2\pi)^6 p_1^0 p_2^0 R^2}(E_1E_2 - p_1p_2\cos\xi)\\ &\times \left|M_{0\nu}^{(\Delta_L)}\right|^2 \delta(E_f + p_1^0 + p_2^0 - E_i)\end{aligned} \tag{4.108}$$

$$\begin{aligned}\left|\langle f|\,\mathcal{A}_{eff}^{\Delta_R}\,|i\rangle\right|^2 =&(Y_{ee})^2 (\frac{m_{W_L}}{m_{W_R}})^8 m_e^2 m_p^2 \frac{1}{\nu_R^4}\frac{2g_A^4 G_F^4 V_{ud}^4}{(2\pi)^6 p_1^0 p_2^0 R^2}(E_1E_2 - p_1p_2\cos\xi)\\ &\times \left|M_{0\nu}^{(\Delta_R)}\right|^2 \delta(E_f + p_1^0 + p_2^0 - E_i)\end{aligned} \tag{4.109}$$

the differential decay rates are

$$\begin{aligned}d\Gamma_{\Delta_L}^{0\nu} =&\left|M_{0\nu}^{(\Delta_L)}\right|^2 (Y_{ee})^2 m_e^2 m_p^2 \frac{1}{\nu_R^4}\frac{\nu_L^2}{\nu_R^2}\frac{4g_A^4 G_F^4 V_{ud}^4}{(2\pi)^5 R^2}(E_1E_2 - p_1p_2\cos\xi)\\ &\times F(Z+2,E_1)F(Z+2,E_2)p_1p_2\sin\xi d\xi dE_2\end{aligned} \tag{4.110}$$

$$\begin{aligned}d\Gamma_{\Delta_R}^{0\nu} =&(\frac{m_{W_L}}{m_{W_R}})^8 m_e^2 m_p^2 \left|M_{0\nu}^{(\Delta_R)}\right|^2 (Y_{ee})^2 \frac{1}{\nu_R^4}\frac{4g_A^4 G_F^4 V_{ud}^4}{(2\pi)^5 R^2}(E_1E_2 - p_1p_2\cos\xi)\\ &\times F(Z+2,E_1)F(Z+2,E_2)p_1p_2\sin\xi d\xi dE_2\end{aligned} \tag{4.111}$$

In assumption that ν_R is really large, the charged Higgs scalar processes are suppressed because of the very small coefficient $\frac{1}{\nu_R^4}$. If we consider the assumption $\nu_R \gg \nu_L$ which is used in discussing the smallness mass of the light neutrino, the decay rate of left-handed charged Higgs mechanism will be really small and thus the Δ_L will be suppressed. In the assumption where $m_{W_R} \gg m_{W_L}$, the right-handed Higgs scalar process is suppressed due to the very small front factor.

4.4 Phase Space Factors and Nuclear Matrix Elements

Let us reorganize how we can calculate the differential decay rate. From the Fermi Golden rule we aware that the decay rate is proportional to the integral of the transition amplitude. When the initial and final states are settled, we can play around of any intermediate states that are allowed. However, we need to sum over all the possible amplitudes in order to obtain a total rate between specific initial and final states. In neutrinoless double beta decay with six fermions interaction

u, u, d, d, e, e in the mLRSM, we could have light, heavy, light-heavy mixing, and Higgs scalars in the middle states. This allows us to write the total amplitude $\mathcal{A}$

$$\mathcal{A}^{0\nu} = \mathcal{A}^{(LL)} + \mathcal{A}^{(RR)} + \mathcal{A}^{(LR)} + \mathcal{A}^{\Delta_L} + \mathcal{A}^{\Delta_R} \tag{4.112}$$

the transition amplitude square will be

$$\begin{aligned}
\left|\langle f|\,\mathcal{A}^{0\nu}\,|i\rangle\right|^2 =& \left|\langle f|\,\mathcal{A}^{(LL)}\,|i\rangle\right|^2 + \left|\langle i|\,\mathcal{A}^{(LL)\dagger}\,|f\rangle\,\langle f|\,\mathcal{A}^{(RR)}\,|i\rangle\right| \\
&+\left|\langle i|\,\mathcal{A}^{(LL)\dagger}\,|f\rangle\,\langle f|\,\mathcal{A}^{(LR)}\,|i\rangle\right| + \left|\langle i|\,\mathcal{A}^{(LL)\dagger}\,|f\rangle\,\langle f|\,\mathcal{A}^{\Delta_L}\,|i\rangle\right| \\
&+\left|\langle i|\,\mathcal{A}^{(LL)\dagger}\,|f\rangle\,\langle f|\,\mathcal{A}^{\Delta_R}\,|i\rangle\right| + \left|\langle f|\,\mathcal{A}^{(RR)}\,|i\rangle\right|^2 \\
&+\left|\langle i|\,\mathcal{A}^{(RR)\dagger}\,|f\rangle\,\langle f|\,\mathcal{A}^{(LR)}\,|i\rangle\right| + \left|\langle i|\,\mathcal{A}^{(RR)\dagger}\,|f\rangle\,\langle f|\,\mathcal{A}^{\Delta_L}\,|i\rangle\right| \\
&+\left|\langle i|\,\mathcal{A}^{(RR)\dagger}\,|f\rangle\,\langle f|\,\mathcal{A}^{\Delta_R}\,|i\rangle\right| + \left|\langle f|\,\mathcal{A}^{(LR)}\,|i\rangle\right|^2 \\
&+\left|\langle i|\,\mathcal{A}^{(LR)\dagger}\,|f\rangle\,\langle f|\,\mathcal{A}^{\Delta_L}\,|i\rangle\right| + \left|\langle i|\,\mathcal{A}^{(LR)\dagger}\,|f\rangle\,\langle f|\,\mathcal{A}^{\Delta_R}\,|i\rangle\right| \\
&+\left|\langle f|\,\mathcal{A}^{\Delta_L}\,|i\rangle\right|^2 + \left|\langle i|\,\mathcal{A}^{\Delta_L\dagger}\,|f\rangle\,\langle f|\,\mathcal{A}^{\Delta_R}\,|i\rangle\right| + \left|\langle f|\,\mathcal{A}^{\Delta_R}\,|i\rangle\right|^2
\end{aligned} \tag{4.113}$$

The terms have only one type of cases of the amplitude such as $\left|\langle f|\,\mathcal{A}^{(LL)}\,|i\rangle\right|^2$ which are the pure terms. The terms with two types of amplitudes are called interference terms, e.g. $\left|\langle i|\,\mathcal{A}^{(LL)\dagger}\,|f\rangle\,\langle f|\,\mathcal{A}^{(RR)}\,|i\rangle\right|$. In this thesis, we calculate the effective amplitude of all the pure terms. One could use the same argument as in the thesis to calculate the rest coupled terms. This can be done by continuing researches.

Phase space factors

We can now write down the phase space factor for each pure case with differential decay rate we have come by. Together With the standard mechanism (4.51) and Equations (4.87), (4.101), (4.104), (4.110) and (4.111), the phase space factors are

$$\begin{aligned}
G^{0\nu}_{(LL)}(Q,Z) =& \frac{G_F^4 V_{ud}^4 g_A^4 m_e^2}{2\ln 2(2\pi)^5 R^2} \int_0^Q dT_1 \int_0^\pi \sin\xi d\xi (E_1E_2 - p_1p_2\cos\xi)p_1p_2 \\
&\times F(Z+2,E_1)F(Z+2,E_2)
\end{aligned} \tag{4.114}$$

$$\begin{aligned}
G^{0\nu}_{(RR)}(Q,Z) =& \frac{G_F^4 V_{ud}^4 g_A^4 m_e^2}{2\ln 2(2\pi)^5 R^2} \int_0^Q dT_1 \int_0^\pi \sin\xi d\xi (E_1E_2 - p_1p_2\cos\xi)p_1p_2 \\
&\times F(Z+2,E_1)F(Z+2,E_2)
\end{aligned} \tag{4.115}$$

$$\begin{aligned}
G^{0\nu}_{(LR-\lambda)}(Q,Z) =& \frac{G_F^4 V_{ud}^4 g_A^4}{2\ln 2(2\pi)^5 R^2} \int_0^Q dT_1 \int_0^\pi \sin\xi d\xi (E_1 - E_2)^2(E_1E_2 - m_e^2 + p_1p_2\cos\xi) \\
&\times p_1p_2F(Z+2,E_1)F(Z+2,E_2)
\end{aligned} \tag{4.116}$$

$$G^{0\nu}_{(LR-\eta)}(Q,Z) = \frac{G_F^4 V_{ud}^4 g_A^4}{2\ln 2(2\pi)^5 R^2} \int_0^Q dT_1 \int_0^\pi \sin\xi d\xi (E_1+E_2)^2 (E_1E_2 - m_e^2 + p_1p_2\cos\xi) \\ \times p_1p_2 F(Z+2,E_1)F(Z+2,E_2) \tag{4.117}$$

$$G^{0\nu}_{(\Delta_L)}(Q,Z) = \frac{2G_F^4 V_{ud}^4 g_A^4 m_e^2}{\ln 2(2\pi)^5 R^2} \int_0^Q dT_1 \int_0^\pi \sin\xi d\xi (E_1E_2 - p_1p_2\cos\xi) p_1p_2 \\ \times F(Z+2,E_1)F(Z+2,E_2) \tag{4.118}$$

$$G^{0\nu}_{(\Delta_R)}(Q,Z) = \frac{2G_F^4 V_{ud}^4 g_A^4 m_e^2}{\ln 2(2\pi)^5 R^2} \int_0^Q dT_1 \int_0^\pi \sin\xi d\xi (E_1E_2 - p_1p_2\cos\xi) p_1p_2 \\ \times F(Z+2,E_1)F(Z+2,E_2) \tag{4.119}$$

where we redefine the phase space factors in LL and RR case with additional m_e^2 to make the unit in each phase space factors identical. The the Phase space factors have the dimension of t^{-1}. We can derive this in the LL case as an example, G_F has the unit of the mass m^{-2}, R has m^{-1}, and the integral provides m^5. Thus the total unit in (4.114) is $[m^{-8}m^2m^2m^5] = [m] = [t^{-1}]$. The half-life without the interference terms proposed in (4.113) is expressed as

$$(T_{\frac{1}{2}})^{-1} = \frac{|m_{\beta\beta}|^2}{m_e^2}\left|M_{0\nu}^{(LL)}\right|^2 G^{0\nu}_{(LL)} + \frac{m_p^2}{|\langle m_N\rangle|^2}(\frac{m_{W_L}}{m_{W_R}})^8 \left|M_{0\nu}^{(RR)}\right|^2 G^{0\nu}_{(RR)} \\ + \sin^2\theta_\nu (\frac{m_{W_L}}{m_{W_R}})^4 \left|M_{0\nu}^{(LR)}\right|^2 G^{0\nu}_{(LR-\lambda)} + \sin^2\xi_W \sin^2\theta_\nu \left|M_{0\nu}^{(LR-\eta)}\right|^2 G^{0\nu}_{(LR-\eta)} \\ + |Y_{ee}|^2 \frac{m_p^2}{\nu_R^4}\frac{\nu_L^2}{\nu_R^2}\left|M_{0\nu}^{(\Delta_L)}\right|^2 G^{0\nu}_{(\Delta_L)} + (\frac{m_{W_L}}{m_{W_R}})^8 |Y_{ee}|^2 \frac{m_p^2}{\nu_R^4}\left|M_{0\nu}^{(\Delta_R)}\right|^2 G^{0\nu}_{(\Delta_R)} \tag{4.120}$$

where we write the phase space factor as $G \equiv G(Q,Z)$ for simplification. One can also write the Yukawa coupling Y_{ee} as the heavy neutrino mass m_{N_i} [12], see section 3.5. However, we will keep the Yukawa coupling here. We can see that the coefficients match with the coefficients in the effective Lagrangian (4.77). The coefficients of each term can be constrained by comparing the experimental limit of the half-life with the theoretical result of NMEs and phase space factors [12]. We can calculate the phase space factors numerically from Equations (4.114-4.119). The NMEs are not calculated in the thesis, so we would take from the literature [106]. In order to take the NME numerical result, we should redefine our NMEs and phase space factors to match the form in the literature. The half-life expression will not change, i.e. we only define the NMEs with/without some constant numbers by dividing/multiplying the numbers to the phase space factors. After values of NMEs and phase space factors are obtained, we can keep only one of the term in (4.120) and force other terms to zero to get the upper limit of the coefficients. The upper

NME		PSF		Coupling	
$M_{0\nu}^{(LL)}$	3.12	$G_{(LL)}^{0\nu}$	0.96	$\lvert m_{\beta\beta}\rvert$	$< 0.19\,\text{eV}$
$M_{0\nu}^{(RR)}$	213.13	$G_{(RR)}^{0\nu}$	0.96	ϵ_{RR}	$< 4.60 \times 10^{-8}$
$M_{0\nu}^{(LR)}$	3.12	$G_{(LR-\lambda)}^{0\nu}$	5.20	λ	$< 1.64 \times 10^{-7}$
$M_{0\nu}^{(LR-\eta)}$	2.00	$G_{(LR-\eta)}^{0\nu}$	40.6	η	$< 7.31 \times 10^{-8}$
$M_{0\nu}^{(\Delta_L)}$	213.13	$G_{\Delta_L)}^{0\nu}$	3.85	ϵ_{Δ_L}	$< 2.30 \times 10^{-8}$
$M_{0\nu}^{(\Delta_R)}$	213.13	$G_{\Delta_R)}^{0\nu}$	3.85	ϵ_{Δ_R}	$< 2.30 \times 10^{-8}$

Table 4.2: Numerical results of NMEs, phase space factors (PSF), and couplings for ^{136}Xe. We have set the component of the CKM matrix $V_{ud} \equiv \cos\theta_C$ to 1. The original numerical value of $M_{0\nu,GT}$ and $M_{0\nu,F}$ are taken from this paper [106]. The NMEs and the couplings are dimensionless, and the unit of phase space factors is $10^{-14} yr^{-1}$, where yr refers to years.

limit indicates how possible the mechanism will take place in $0\nu\beta\beta$ process. In the standard mechanism coupling ϵ_{LL} in (4.79), if the relation $V_{ud} = U_{ee} = 1$ is taken [68, 108], we can obtain a upper limit of the mass of the electron neutrino. The numerical results of ^{136}Xe are in Table 4.2.

Nuclear matrix element

The NME is usually complicated to calculate analytically and needs to be obtained from numerical simulations and sensitivity experiments [109, 110]. In this thesis, we only consider the S-wave emitted electron wave functions $S_{\frac{1}{2}}$, i.e. the first order contributions. The zero spin change of the S-wave suggests the only $0^+ \to 0^+$ transition, where the hadrons contain only vector and axial vector currents. In a more precise derivation, the P wave $P_{\frac{1}{2}}$ is also considered as the second order for $0^+ \to 0^+$, where we have the hadronic currents with the different combinations of the emitted electron wave functions [89].

$$\begin{aligned}
\sqrt{2}S_1S_2 &: g_V^2 - g_A^2\vec{\sigma}_1\cdot\vec{\sigma}_2 \\
2S_1S_2 &: g_V^2 \mp g_A^2\vec{\sigma}_1\cdot\vec{\sigma}_2 \\
(S_1P_2 - P_1S_2)r_{12}/R &: g_V^2 \pm g_A^2(\frac{1}{3}\vec{\sigma}_1\cdot\vec{\sigma}_2 - 2S_{12}) \\
(S_1P_2 + P_1S_2)r_{+12}/R &: g_Vg_A(\vec{\sigma}_1 - \vec{\sigma}_2)\cdot(\hat{r}_{12}\times\hat{r}_{+12}) \\
2S_1S_2 &: g_Vg_A\hat{r}_{12}\cdot(\vec{\sigma}\times D_2 - \vec{\sigma}\times D_1)
\end{aligned} \tag{4.121}$$

where S_i and P_i denote the electron radial wave functions of $S_{\frac{1}{2}}$ and $P_{\frac{1}{2}}$ respectively. These wave functions are combinations of solutions of the Dirac equation of outgoing electron wave functions with specific nuclear potential [89, 90]. The notation is $\hat{r} \equiv \frac{\vec{r}}{|\vec{r}|}$. The vector operators are defined as

$$\begin{aligned} \vec{r}_{+nm} =& \vec{r}_n + \vec{r}_m \\ S_{nm} =& (\vec{\sigma}_n \cdot \hat{r}_{nm})(\vec{\sigma}_m \cdot \hat{r}_{nm}) - \frac{1}{3}\vec{\sigma}_n \cdot \vec{\sigma}_m \end{aligned} \tag{4.122}$$

D is the nuclear recoil current,

$$\begin{aligned} D_n =& V_n^{(1)} + (\frac{g_W}{g_V})W_n^{(0)} \\ =& \frac{\vec{p}_n + \vec{p}'_n - i\mu_\beta \vec{\sigma}_n \times (\vec{p}_n - \vec{p}'_n)}{2M} \end{aligned} \tag{4.123}$$

where $\mu_\beta = \kappa_\beta + 1$, and $\kappa_\beta = 3.70$ is the isovector anomalous magnetic moment of the nucleon. V and W stands for the vector current and the weak magnetism current, respectively. A well established method for NME calculations is the QRPA (quasiparticle random phase approximation), see References [106, 111, 112].

4.5 Energy Spectra and Angular Correlations

The single electron kinetic energy spectra and the angular correlation are two useful measurable quantities in $0\nu\beta\beta$ decay [113]. They can be used to classify the decay processes and the contributions of theories [114]. Let us explicitly derive the formula. We can divide a dE_2 on both sides of the differential decay rate equations for each case and integrate the correlation angle. After the integration, there will be only one parameter which is one of the electron energy. On the other hand, one can integrate the differential decay rate with respect to the energy and the correlation angle then perform a derivative with respect to the energy to obtain a energy depend differential decay rate spectrum. Let us take the standard mass mechanism cases for an instance. Divide dE_2 by Equation (4.49) and integrate the angle ξ,

$$\frac{d\Gamma^{0\nu}_{(LL)}}{dE_2} = |m_{\beta\beta}|^2 \left| M^{(LL)}_{0\nu} \right|^2 \frac{G_F^4 V_{ud}^4 g_A^4}{(\pi)^5 R^2} F(Z+2, E_1) F(Z+2, E_2) 2E_1 E_2 p_1 p_2 \tag{4.124}$$

where E_1 can be written in terms of E_2: $E_1 = Q + 2m_e - E_2$ and $E^2 = p^2 + m_e^2$. The NME can be considered as some number which is calculated numerically in some papers [115, 116, 117]. In this case, the coefficient and the NME as well as the neutrino effective mass can be normalized to a factor, and the shape of the spectrum will only depend on the leptonic part,

$$\frac{d\Gamma^{0\nu}_{(LL)}}{dE_2} = C_{LL} F(Z+2, E_1) F(Z+2, E_2) 2E_1 E_2 p_1 p_2 \tag{4.125}$$

where C_{LL} is short for the rest multiples of Equation (4.124). In other cases, We can also rewrite the number parts into factors C_{cases}, and they are

$$\frac{d\Gamma^{0\nu}_{(RR)}}{dE_2} = C_{RR}F(Z+2,E_1)F(Z+2,E_2)2E_1E_2p_1p_2 \tag{4.126}$$

$$\frac{d\Gamma^{0\nu}_{(LR-\lambda)}}{dE_2} = C_{LR-\lambda}F(Z+2,E_1)F(Z+2,E_2)2(E_1-E_2)^2(E_1E_2-m_e^2)p_1p_2 \tag{4.127}$$

$$\frac{d\Gamma^{0\nu}_{(LR-\eta)}}{dE_2} = C_{LR-\eta}F(Z+2,E_1)F(Z+2,E_2)2(E_1+E_2)^2(E_1E_2-m_e^2)p_1p_2 \tag{4.128}$$

$$\frac{d\Gamma^{0\nu}_{(\Delta_L)}}{dE_2} = C_{\Delta_L}F(Z+2,E_1)F(Z+2,E_2)2E_1E_2p_1p_2 \tag{4.129}$$

$$\frac{d\Gamma^{0\nu}_{(\Delta_R)}}{dE_2} = C_{\Delta_R}F(Z+2,E_1)F(Z+2,E_2)2E_1E_2p_1p_2 \tag{4.130}$$

From the above equations we can see that the leptonic parts of the LL, Δ_L, Δ_R cases are equivalent. We now show the single electron kinetic energy spectrum in Figure 4.12 (a)(c)(e) on the left penal. The x-axis is the normalized kinetic energy $(E_2 - m_e)/Q$, where Q is the decay energy. Thus in this case, the x-axis is drawn from 0 to 1 instead of m_e to $Q + m_e$ if we choose E_2 to be the x-axis. In the normalized differential decay rate, only the leptonic parts contribute to the shape of the spectra. In the above five cases, we would only have two different shape of spectra, drawing in Figure 4.12. The shape of the LR-λ case should be different from the other cases due to the factor $(E_1 - E_2)^2$ [113]. Figures 4.12a and 4.12e shows that the two electrons in these cases have a tendency to gain the same masses. However, in the λ-contribution shown in Figure 4.12c, the two electrons are prevented to have the same masses. Next, Let us derive the expression of the angular correlation. The differential decay rate contains the structure $1 \pm \cos\theta$. In the standard mechanism (4.49) for example, the differential decay rate contains two parts $E_1E_2 - p_1p_2\cos\xi$ that only one part contain the electron coupling angle ξ. We can define two coefficients as a_0 and a_1. Then we can rewrite the differential decay rate as

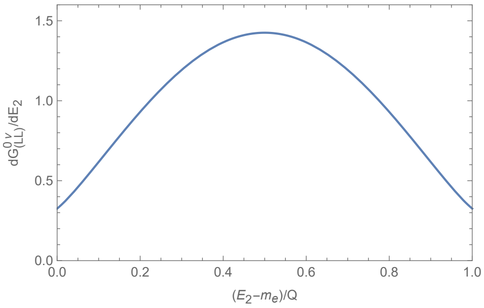
$$\frac{d\Gamma^{0\nu}}{d(\cos\xi)dE_2} = a_0(1 + \frac{a_1}{a_0}\cos\xi) \tag{4.131}$$

where we put all the coefficients and NMEs in a_0, a_1 in each case. The ratio $\frac{a_1}{a_0}$ which only includes the difference between the $\cos\xi$ term and the no-angle term is defined as the angular correlation factor. For example, in the LL case (4.49) and the LR case (4.101) (4.104), this factor is

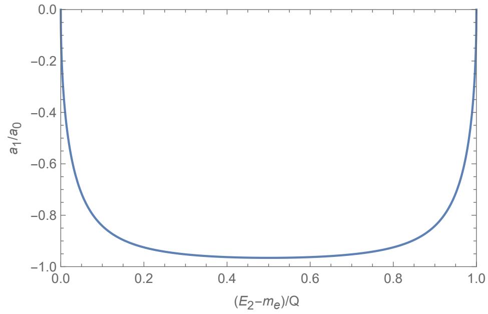
$$LL: \frac{a_1}{a_0} = -\frac{p_1 p_2}{E_1 E_2} \tag{4.132}$$

$$LR: \frac{a_1}{a_0} = \frac{p_1 p_2}{E_1 E_2 - m_e^2} \tag{4.133}$$

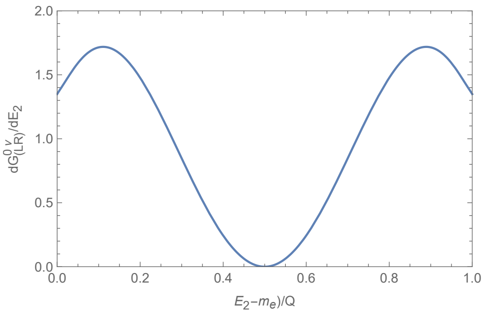
respectively. The angular correlation factor in the RR case and in the charged Higgs scalar cases (Δ_L,Δ_R cases) are exactly the same as the one in the standard mechanism (LL case), i.e. Equation 4.132. The λ-contribution and η-contribution have the same angular correlation, Equation (4.133). The angular correlation figures are presented in the right panel of Figure 4.12. We also take the normalized kinetic energy as the x-axis. The negative angular correlation as well as Figure 4.12b show that the two electrons tend to be emitted in the opposite direction in these cases, while the two electrons in LR cases with positive angular correlation are tend to be emitted in the same direction. However, the angular correlation in LR cases maybe not hold in more precise approximation. In the λ-Contribution, the electron wave function we took is the plane wave function, and only the first term in the expansions was taken into account, i.e. $\mathrm{e}^{-i\vec{p}_1\cdot\vec{x}_1 - i\vec{p}_2\cdot\vec{x}_2} \approx 1$. This rough approximation gives the same angular correlation shape in λ and η cases. One could also use a more precise approximation that takes the first two orders: $\mathrm{e}^{-i\vec{p}_1\cdot\vec{x}_1 - i\vec{p}_2\cdot\vec{x}_2} \approx 1 - \vec{p}_1\cdot\vec{x}_1 - \vec{p}_2\cdot\vec{x}_2$, which will provide different angular correlation spectra in the two contributions due to the factors $(E_1 - E_2)^2$ and $(E_1 + E_2)^2$ [87, 118].



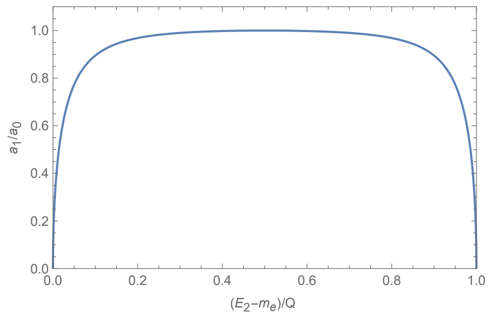
(a) "standard case (also for RR, Δ_L, and Δ_R cases)" spectrum



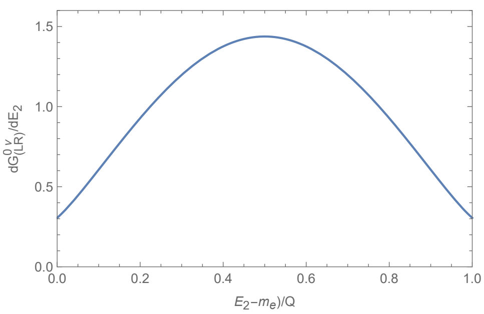
(b) "standard case (also for RR, Δ_L, and Δ_R cases)" angular correlation



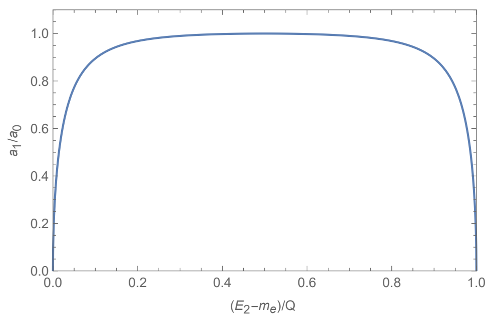
(c) "LR-λ" case spectrum



(d) "LR-λ" case angular correlation



(e) "LR-η" case spectrum



(f) "LR-η" case angular correlation

Figure 4.12: (a)(c)(e): normalized differential decay rate spectra. The shapes of the spectra of LL, RR, LR-η, Δ_L, and Δ_R are identical. The only difference is the C_{cases} value before the normalization. (b)(d)(f): angular correlation factor. The angular correlation of the λ-Contribution and the η-Contribution are the same in the zeroth order approximation of electron wave functions $\mathrm{e}^{-i\vec{p}_1\cdot\vec{x}_1-i\vec{p}_2\cdot\vec{x}_2} \approx 1$. If the higher orders of the expansion series are considered, we will have different angular correlations in λ and η cases, see [87, 113].

5 Summary and Conclusion

In this thesis, we have studied the mLRSM from model building and calculated the differential decay rate of the $0\nu\beta\beta$ decay under the mLRSM. The mLRSM has been naturally introduced when we encountered the right-handed currents and considered parity restoration at some high-energy scale. This model gives an explanation for the smallness of the neutrino mass from the type I+II seesaw mechanism. The charged leptons masses are proportional to the VEVs of the Higgs bidoublet (see Equation (3.66)). The light neutrino mass can be related to the VEVs of the Higgs bidoublet divided by the heavy Higgs triplet VEV, see Equation (3.86). Thus, this possibly indicates the relation of the light neutrino (3.88), which provides a not so large right-handed W boson that could be observed from experiments.

$0\nu\beta\beta$ being a lepton number violation process is a very practical way to determine the nature of the neutrinos. The probability amplitude explicitly depends on the underlying mechanism of $0\nu\beta\beta$ decay. Therefore, it is worth calculating the differential decay rate. We have calculated the differential decay rate in the mLRSM in chapter 4. This decay rate calculation is under the low-energy effective approximation, where we supposed the propagating energy to be $\mathcal{O}(100\text{MeV}) \ll m_W$. We have shown all the possible tree-level contributions in the mLRSM graphically in Figure 4.3, and from Figure 4.5 to Figure 4.9. The amplitudes of each process have been given in section 4.3 by using the Feynman rules given in Appendix E. Then we derived the differential decay rates and the total decay rates from the transition probabilities. These expressions can be used to put constrains on the single interaction coefficients. The numerical values of the phase space factors have been calculated and are presented in the second column of Table 4.2. We matched the mLRSM onto the effective low-energy Lagrangian in Equation (4.77). The corresponding low-energy coupling constants are given in Equation (4.79). The numerical values of the upper limit of these couplings, in the reference to the latest data from KamLand-Zen, are given in the third column of Table 4.2. The expressions of the couplings, e.g. in (4.79), depend on the mechanism and the physical models. Therefore, we are planning to research more proper models and to look for the underlying mechanism of the BSM in the near future.

We have drawn the single electron kinetic energy spectra and the angular correlation figures in Figure 4.12. The different shape of the λ-contribution in the energy spectra is due to the factor $(E_1 - E_2)^2$. The Figure 4.12a and 4.12e shows that the two out-going electrons in standard mass mechanism and η-case are more likely to possess the same energy, while in the λ-case Figure 4.12c, shows that the two electrons tend to have different energies. The angular correlation of the chirality exchange cases (LR cases) has the $(1+\cos\xi)$ type while the other cases have $(1-\cos\xi)$

type correlation, which gives the shape of LR cases (Figure 4.12d and 4.12f) differ from the standard mechanism (Figure 4.12b). It follows that in the LR case, which features a positive angular correlation, the two electrons tend to be emitted in the same direction while in the other cases, the two electrons tend to be emitted in the opposite direction due to the negative angular correlation.

There are also some aspects that can be improved in a future study. The interference terms of different diagrams in (4.113) can be calculated in order to obtain a full half-life expression. The electron wave functions, instead of using the plan wave approximation, can be solved directly from the Dirac equation. However, the Dirac equation is difficult to solve analytically, instead, we can solve them numerically through programming.

Part I

Appendix

A Explicit Higgs Potential in mLRSM

The full Higgs potential in Equation (3.2) would be (one can also find some other version here [59, 63, 64, 73],

$$
\begin{aligned}
V(\phi,\tilde{\phi},\Delta_L,\Delta_R) = & -\mu_{11}^2\mathrm{Tr}[\phi^\dagger\phi] - \mu_{12}^2\mathrm{Tr}[\phi^\dagger\tilde{\phi}+\tilde{\phi}^\dagger\phi] + \lambda_{1111}(\mathrm{Tr}[\phi^\dagger\phi])^2 \\
& + \lambda_{1211}\mathrm{Tr}[\phi^\dagger\tilde{\phi}+\tilde{\phi}^\dagger\phi]\mathrm{Tr}[\phi^\dagger\phi] + \lambda_{1212}\big\{(\mathrm{Tr}[\phi^\dagger\tilde{\phi}])^2+(\mathrm{Tr}[\tilde{\phi}^\dagger\phi])^2\big\} \\
& + \lambda_{1221}\mathrm{Tr}[\phi^\dagger\tilde{\phi}]\mathrm{Tr}[\tilde{\phi}^\dagger\phi] + \lambda'_{1111}\mathrm{Tr}[\phi^\dagger\phi\phi^\dagger\phi] \\
& + \lambda'_{1211}(\mathrm{Tr}[\phi^\dagger\tilde{\phi}\phi^\dagger\phi]+\mathrm{Tr}[\phi^\dagger\phi\tilde{\phi}^\dagger\phi]) + \lambda'_{1122}\mathrm{Tr}[\phi^\dagger\phi\tilde{\phi}^\dagger\tilde{\phi}] \\
& - \mu^2\mathrm{Tr}[\Delta_L^\dagger\Delta_L+\Delta_R^\dagger\Delta_R] + \rho_1\big\{(\mathrm{Tr}[\Delta_L^\dagger\Delta_L])^2+(\mathrm{Tr}[\Delta_R^\dagger\Delta_R])^2\big\} \\
& + \rho_2\big\{\mathrm{Tr}[\Delta_L\Delta_L]\mathrm{Tr}[\Delta_L^\dagger\Delta_L^\dagger]+\mathrm{Tr}[\Delta_R\Delta_R]\mathrm{Tr}[\Delta_R^\dagger\Delta_R^\dagger]\big\} \\
& + \rho_3\mathrm{Tr}[\Delta_L^\dagger\Delta_L]\mathrm{Tr}[\Delta_R^\dagger\Delta_R] \\
& + \rho_4\big\{\mathrm{Tr}[\Delta_L\Delta_L]\mathrm{Tr}[\Delta_R^\dagger\Delta_R^\dagger]+\mathrm{Tr}[\Delta_R\Delta_R]\mathrm{Tr}[\Delta_L^\dagger\Delta_L^\dagger]\big\} \\
& + \alpha_{11}\mathrm{Tr}[\phi^\dagger\phi](\mathrm{Tr}[\Delta_L^\dagger\Delta_L]+\mathrm{Tr}[\Delta_R^\dagger\Delta_R]) \\
& + \alpha_{22}\mathrm{Tr}[\tilde{\phi}^\dagger\tilde{\phi}](\mathrm{Tr}[\Delta_L^\dagger\Delta_L]+\mathrm{Tr}[\Delta_R^\dagger\Delta_R]) \\
& + \alpha_{12}\mathrm{Tr}[\phi^\dagger\tilde{\phi}+\tilde{\phi}^\dagger\phi](\mathrm{Tr}[\Delta_L^\dagger\Delta_L]+\mathrm{Tr}[\Delta_R^\dagger\Delta_R]) \\
& + \beta_{11}(\mathrm{Tr}[\Delta_L^\dagger\Delta_L\phi\phi^\dagger]+\mathrm{Tr}[\Delta_R^\dagger\Delta_R\phi^\dagger\phi]) \\
& + \beta_{12}(\mathrm{Tr}[\Delta_L^\dagger\Delta_L\phi\tilde{\phi}^\dagger]+\mathrm{Tr}[\Delta_R^\dagger\Delta_R\tilde{\phi}^\dagger\phi]+\mathrm{Tr}[\Delta_L^\dagger\Delta_L\phi^\dagger\tilde{\phi}]+\mathrm{Tr}[\Delta_R^\dagger\Delta_R\tilde{\phi}\phi^\dagger]) \\
& + \beta_{22}(\mathrm{Tr}[\Delta_L^\dagger\Delta_L\tilde{\phi}\tilde{\phi}^\dagger]+\mathrm{Tr}[\Delta_R^\dagger\Delta_R\tilde{\phi}^\dagger\tilde{\phi}])] \\
& + \gamma_{11}(\mathrm{Tr}[\Delta_L^\dagger\phi\Delta_R\phi^\dagger]+\mathrm{Tr}[\Delta_R^\dagger\phi\Delta_L\phi^\dagger]) \\
& + \gamma_{12}(\mathrm{Tr}[\Delta_L^\dagger\phi\Delta_R\tilde{\phi}^\dagger]+\mathrm{Tr}[\Delta_L^\dagger\tilde{\phi}\Delta_R\phi^\dagger]) \\
& + \gamma_{22}(\mathrm{Tr}[\Delta_L^\dagger\tilde{\phi}\Delta_R\tilde{\phi}^\dagger]+\mathrm{Tr}[\Delta_R^\dagger\tilde{\phi}\Delta_L\tilde{\phi}^\dagger])
\end{aligned} \tag{A.1}
$$

where the coefficients satisfy the following relations to ensure the Hermicity and left-right symmetry, also we change some of the subscript to simplify the writing,

$$
\begin{array}{ccc}
\mu_{12}=\mu_{21}\equiv\mu_2 & \lambda_{1111}\equiv\lambda_1 & \lambda_{1211}\equiv\lambda_2 \\
\lambda_{1212}=\lambda_{2121}\equiv\lambda_3 & \lambda_{1221}\equiv\lambda_4 & \lambda'_{1111}\equiv\lambda'_1 \\
\lambda'_{1211}=\lambda'_{1121}\equiv\lambda_2 & \lambda'_{1122}\equiv\lambda'_3 & \alpha_{11}\equiv\alpha_1 \\
\alpha_{12}=\alpha_{21}\equiv\alpha_2 & \alpha_{22}\equiv\alpha_3 & \beta_{11}\equiv\beta_1 \\
\beta_{12}=\beta_{21}\equiv\beta_2 & \beta_{22}\equiv\beta_3 & \gamma_{11}\equiv\gamma_1 \\
\gamma_{12}=\gamma_{21}\equiv\gamma_2 & \gamma_{22}\equiv\gamma_3 &
\end{array} \tag{A.2}
$$

Some equivalent terms have been neglected in Equation (A.1). For example, consider the terms $\mathrm{Tr}[\phi^\dagger\phi]$ and $\mathrm{Tr}[\tilde{\phi}^\dagger\tilde{\phi}]$, for each terms they are Hermitian and can be regarded as different single terms. However, because of the trace property, we can see: $\mathrm{Tr}[(\sigma_2\phi^*\sigma_2)^\dagger(\sigma_2\phi^*\sigma_2)] = \mathrm{Tr}[\sigma_2\phi^T\sigma_2\sigma_2\phi^*\sigma_2] = \mathrm{Tr}[\phi^T\phi^*] = \mathrm{Tr}[(\phi^\dagger\phi)^*]$, if $\phi^\dagger\phi$ leaves real, then we have $\mathrm{Tr}[\tilde{\phi}^\dagger\tilde{\phi}] = \mathrm{Tr}[\phi^\dagger\phi]$. More practical, if we insert the VEVs,

$$\langle\phi\rangle^\dagger\langle\phi\rangle = \begin{pmatrix} \kappa^* & 0 \\ 0 & \kappa'^* \end{pmatrix}\begin{pmatrix} \kappa & 0 \\ 0 & \kappa' \end{pmatrix} = \begin{pmatrix} |\kappa|^2 & 0 \\ 0 & |\kappa'|^2 \end{pmatrix} \tag{A.3}$$

which is automatically left real. It is obvious to see from the above that the potential V is complicated in general writing. Thus, it is better to insert the VEVs first and then discuss which terms are equivalent. Moreover, since we only need the diagonal elements of the matrices, the VEVs of Higgs bidoublets are rigorously used to tell if two terms are equivalent. However, the hermicity and parity conjugate symmetry should be discussed at the first place, i.e. we need to keep the potential Hermitian and left-right symmetric. For instance, the term $-\mu_{12}^2\mathrm{Tr}[\phi^\dagger\tilde{\phi} + \tilde{\phi}^\dagger\phi]$ is Hermitian, but the single term $(\phi^\dagger\tilde{\phi})^\dagger = \tilde{\phi}^\dagger\phi \neq \phi^\dagger\tilde{\phi}$ is indeed not Hermitian.

B Explicit Calculation of Higgs Kinetic Terms

The explicit derivation of this interaction from the kinetic term is

$$
\begin{aligned}
&\mathrm{Tr}[(D_\mu\Delta_L)^\dagger(D^\mu\Delta_L)] + \mathrm{Tr}[(D_\mu\Delta_R)^\dagger(D^\mu\Delta_R)]\\
=&\mathrm{Tr}\big[\big(\partial_\mu \begin{pmatrix} \frac{1}{\sqrt{2}}\delta_L^+ & \delta_L^{++} \\ \delta_L^0 & -\frac{1}{\sqrt{2}}\delta_L^+ \end{pmatrix}\\
&+\frac{ig}{2}\begin{pmatrix} W_{1L,\mu}(\delta_L^0-\delta_L^{++})-iW_{2L,\mu}(\delta_L^0+\delta_L^{++}) & -\frac{2}{\sqrt{2}}\delta_L^+(W_{1L,\mu}-iW_{2L,\mu}) \\ \frac{2}{\sqrt{2}}\delta_L^+(W_{1L,\mu}+iW_{2L,\mu}) & -W_{1L,\mu}(\delta_L^0-\delta_L^{++})+iW_{2L,\mu}(\delta_L^0+\delta_L^{++}) \end{pmatrix}\\
&+ig'B_\mu \begin{pmatrix} \frac{1}{\sqrt{2}}\delta_L^+ & \delta_L^{++} \\ \delta_L^0 & -\frac{1}{\sqrt{2}}\delta_L^+ \end{pmatrix}\big)^\dagger\\
&\times\big(\partial^\mu \begin{pmatrix} \frac{1}{\sqrt{2}}\delta_L^+ & \delta_L^{++} \\ \delta_L^0 & -\frac{1}{\sqrt{2}}\delta_L^+ \end{pmatrix}\\
&+\frac{ig}{2}\begin{pmatrix} W^\mu_{1L}(\delta_L^0-\delta_L^{++})-iW^\mu_{2L}(\delta_L^0+\delta_L^{++}) & -\frac{2}{\sqrt{2}}\delta_L^+(W^\mu_{1L}-iW^\mu_{2L}) \\ \frac{2}{\sqrt{2}}\delta_L^+(W^\mu_{1L}+iW^\mu_{2L}) & -W^\mu_{1L}(\delta_L^0-\delta_L^{++})+iW^\mu_{2L}(\delta_L^0+\delta_L^{++}) \end{pmatrix}\\
&+ig'B^\mu \begin{pmatrix} \frac{1}{\sqrt{2}}\delta_L^+ & \delta_L^{++} \\ \delta_L^0 & -\frac{1}{\sqrt{2}}\delta_L^+ \end{pmatrix}\big)\big]\\
+&\mathrm{Tr}\big[\big(\partial_\mu \begin{pmatrix} \frac{1}{\sqrt{2}}\delta_R^+ & \delta_R^{++} \\ \delta_R^0 & -\frac{1}{\sqrt{2}}\delta_R^+ \end{pmatrix}\\
&+\frac{ig}{2}\begin{pmatrix} W_{1R,\mu}(\delta_R^0-\delta_R^{++})-iW_{2R,\mu}(\delta_R^0+\delta_R^{++}) & -\frac{2}{\sqrt{2}}\delta_R^+(W_{1R,\mu}-iW_{2R,\mu}) \\ \frac{2}{\sqrt{2}}\delta_R^+(W_{1R,\mu}+iW_{2R,\mu}) & -W_{1R,\mu}(\delta_R^0-\delta_R^{++})+iW_{2R,\mu}(\delta_R^0+\delta_R^{++}) \end{pmatrix}\\
&+ig'B_\mu \begin{pmatrix} \frac{1}{\sqrt{2}}\delta_R^+ & \delta_R^{++} \\ \delta_R^0 & -\frac{1}{\sqrt{2}}\delta_R^+ \end{pmatrix}\big)^\dagger\\
&\times\big(\partial^\mu \begin{pmatrix} \frac{1}{\sqrt{2}}\delta_R^+ & \delta_R^{++} \\ \delta_R^0 & -\frac{1}{\sqrt{2}}\delta_R^+ \end{pmatrix}\\
&+\frac{ig}{2}\begin{pmatrix} W^\mu_{1R}(\delta_R^0-\delta_R^{++})-iW^\mu_{2R}(\delta_R^0+\delta_R^{++}) & -\frac{2}{\sqrt{2}}\delta_R^+(W^\mu_{1R}-iW^\mu_{2R}) \\ \frac{2}{\sqrt{2}}\delta_R^+(W^\mu_{1R}+iW^\mu_{2R}) & -W^\mu_{1R}(\delta_R^0-\delta_R^{++})+iW^\mu_{2R}(\delta_R^0+\delta_R^{++}) \end{pmatrix}\\
&+ig'B^\mu \begin{pmatrix} \frac{1}{\sqrt{2}}\delta_R^+ & \delta_R^{++} \\ \delta_R^0 & -\frac{1}{\sqrt{2}}\delta_R^+ \end{pmatrix}\big)\big]
\end{aligned}
\tag{B.1}
$$

we will have 9 terms inside each trace. We can use the trace sum property to trace each term one by one. For example, the partial derivative term is combined as $\partial_\mu \begin{pmatrix} \frac{1}{\sqrt{2}}\delta_L^+ & \delta_L^{++} \\ \delta_L^0 & -\frac{1}{\sqrt{2}}\delta_L^+ \end{pmatrix}^\dagger \partial^\mu \begin{pmatrix} \frac{1}{\sqrt{2}}\delta_L^+ & \delta_L^{++} \\ \delta_L^0 & -\frac{1}{\sqrt{2}}\delta_L^+ \end{pmatrix}$. After calculating the traces of all the terms and summing together, we have the interactions of the left-handed fields

$$
\begin{aligned}
&\mathrm{Tr}[(D_\mu\Delta_L)^\dagger(D^\mu\Delta_L)] \\
=& \partial_\mu\delta_L^+\partial^\mu\delta_L^+ + \partial_\mu\delta_L^0\partial^\mu\delta_L^0 + \partial_\mu\delta_L^{++}\partial_\mu\delta_L^{++} \\
&+ \frac{ig}{\sqrt{2}}\partial_\mu\delta_L^+[W_{1L}^\mu(\delta_L^0 - \delta_L^{++}) - iW_{2L}^\mu(\delta_L^0 + \delta_L^{++})] + \frac{ig}{\sqrt{2}}\delta_L^+\partial_\mu\delta_L^0(W_{1L}^\mu + iW_{2L}^\mu) \\
&- \frac{ig}{\sqrt{2}}\delta_L^+\partial_\mu\delta_L^{++}(W_{1L}^\mu - iW_{2L}^\mu) + ig'\delta_L^{++}B^\mu\partial_\mu\delta_L^{++} + ig'\delta_L^0B^\mu\partial_\mu\delta_L^0 \\
&+ \frac{1}{2}B^\mu[(\partial_\mu\delta_L^{++})\delta_L^+ + (\partial_\mu\delta_L^+)\delta_L^{++}] + \frac{g^2}{2}W_{1L,\mu}W_{1L}^\mu(\delta_L^0 - \delta_L^{++})^2 + \frac{g^2}{2}W_{2L,\mu}W_{2L}^\mu(\delta_L^0 + \delta_L^{++})^2 \\
&+ g^2\delta_L^{+2}(W_{1L,\mu} - iW_{2L,\mu})(W_{1L}^\mu + iW_{2L}^\mu) + \frac{1}{\sqrt{2}}gg'W_{1L,\mu}B^\mu(\delta_L^0 - \delta_L^{++})\delta_L^+ \\
&+ \frac{1}{\sqrt{2}}gg'W_{2L,\mu}B^\mu(\delta_L^0 + \delta_L^{++})\delta_L^+ + \frac{1}{\sqrt{2}}gg'\delta_L^+\delta_L^0(W_{1L,\mu} - iW_{2L,\mu})B^\mu \\
&- \frac{1}{\sqrt{2}}gg'\delta_L^+\delta_L^{++}(W_{1L,\mu} + iW_{2L,\mu})B^\mu - ig'\delta_L^0B_\mu\partial^\mu\delta_L^0 - ig'\delta_L^{++}B_\mu\partial^\mu\delta_L^{++} - ig'\delta_L^+B_\mu\partial^\mu\delta_L^+ \\
&+ \frac{gg'}{\sqrt{2}}\delta_L^+(\delta_L^0 - \delta_L^{++})B_\mu W_{1L}^\mu - \frac{gg'}{\sqrt{2}}\delta_L^+(\delta_L^0 + \delta_L^{++})B_\mu W_{2L}^\mu + \frac{gg'}{\sqrt{2}}\delta_L^+\delta_L^0B_\mu(W_{1L}^\mu + iW_{2L}^\mu) \\
&- \frac{gg'}{\sqrt{2}}\delta_L^+\delta_L^{++}B_\mu(W_{1L}^\mu - iW_{2L}^\mu) + g'^2\delta_L^{+2}B_\mu B^\mu + g'^2{\delta^0}^2_LB_\mu B^\mu + g'^2\delta_L^{++2}B_\mu B^\mu
\end{aligned} \tag{B.2}
$$

The trace of the right-handed Higgs scalar kinetic term is identical in the form as the left-handed one with only the right-handed subscript in difference. Introduce the upper and lower gauge fields $W_{L,R}^\pm = \frac{1}{\sqrt{2}}(W_{1L,R} \pm iW_{2L,R})$ to simplify the equation. If we consider the VEVs ν_L and insert this into the (B.2), we will find the couplings of the interactions, i.e. the amputated vertex expressions. For instance, the coupling of the gauge bosons-Higgs three-leg vertex comes from

$$
\begin{aligned}
&\frac{g^2}{2}W_{1L,\mu}W_{1L}^\mu(\delta_L^0 - \delta_L^{++})^2 + \frac{g^2}{2}W_{2L,\mu}W_{2L}^\mu(\delta_L^0 + \delta_L^{++})^2 \\
=&\frac{g^2}{2}\frac{1}{2}(W_{L\mu}^+W_L^{+,\mu} + W_{L\mu}^-W_L^{-,\mu} + 2W_{L\mu}^+W_L^{-\mu})({\delta_L^0}^2 + \delta_L^{++2} - 2\delta_L^0\delta_L^{++}) \\
-&\frac{g^2}{2}\frac{1}{2}(W_{L\mu}^+W_L^{+,\mu} + W_{L\mu}^-W_L^{-,\mu} - 2W_{L\mu}^+W_L^{-\mu})({\delta_L^0}^2 + \delta_L^{++2} + 2\delta_L^0\delta_L^{++}) \\
=&g^2W_{L\mu}^+W_L^{-\mu}({\delta_L^0}^2 + \delta_L^{++2}) - g^2(W_{L\mu}^+W_L^{+,\mu} + W_{L\mu}^-W_L^{-,\mu})\delta_L^0\delta_L^{++}
\end{aligned} \tag{B.3}
$$

We can collect all the upper and lower terms and decouple to the pure fields compositions in order to obtain the mass terms and vertices. The second term of (B.3) shows that the $W^{\pm}W^{\pm}\delta_L^{++}$ vertex is proportional to $-ig'^2\nu_{L/R}$.

Inserting the VEVs into the traces, we have

$$\mathrm{Tr}[(D_\mu\phi)^\dagger(D^\mu\phi)] = \mathrm{Tr}\left[\frac{g^2}{4}\begin{pmatrix}\diamondsuit & ... \\ ... & \heartsuit\end{pmatrix}\right] \tag{B.4}$$

where $\diamondsuit$ and $\heartsuit$ are equations,

$$\begin{aligned}\diamondsuit =& \kappa^2[(W_{3L}-W_{3R})^2 + 2W_{L,\mu}^{+}W_L^{-,\mu}] - 2\kappa\kappa' W_{L,\mu}^{-}W_R^{+,\mu} \\ &- 2\kappa'\kappa W_{R,\mu}^{-}W_L^{+,\mu} + 2\kappa'^2 W_{R,\mu}^{+}W_R^{-,\mu}\end{aligned} \tag{B.5}$$

$$\begin{aligned}\heartsuit =& \kappa'^2[2W_{L,\mu}^{+}W_L^{-,\mu} + (W_{3R}-W_{3L})^2] - 2\kappa\kappa' W_{R,\mu}^{+}W_L^{-,\mu} \\ &- 2\kappa'\kappa W_{L,\mu}^{+}W_R^{-,\mu} + 2\kappa^2 W_{R,\mu}^{+}W_R^{-,\mu}\end{aligned} \tag{B.6}$$

"..." in the off-diagonal parts do not contribute to the trace, so it is not necessary to express it. This term becomes

$$\begin{aligned}\mathrm{Tr}[(D_\mu\phi)^\dagger(D^\mu\phi)] =& \frac{g^4}{4}(\kappa^2+\kappa'^2)(W_{3L}^\mu - W_{3R}^\mu)^2 + \frac{g^2}{2}(\kappa^2+\kappa'^2)W_{L,\mu}^{+}W_L^{-,\mu} \\ &+ \frac{g^2}{2}(\kappa^2+\kappa'^2)W_{R,\mu}^{+}W_R^{-,\mu} - g^2\kappa\kappa' W_{L,\mu}^{-}W_R^{+,\mu} - g^2\kappa\kappa' W_{R,\mu}^{-}W_L^{+,\mu}\end{aligned} \tag{B.7}$$

Similarly, we have for the second kinetic term,

$$\begin{aligned}\mathrm{Tr}[(D_\mu\Delta_L)^\dagger(D^\mu\Delta_L)] = \mathrm{Tr}\Bigg[&\frac{g^2}{4}\begin{pmatrix}({W_{1L}}^2+{W_{2L}}^2){\nu_L}^2 + 4{\nu_L}^2{W_{3L}}^2 & ... \\ ... & {\nu_L}^2({W_{1L}}^2+{W_{2L}}^2)\end{pmatrix} \\ &+ \frac{X}{4}gg'\begin{pmatrix}-2W_{3L}^\mu{\nu_L}^2 & ... \\ ... & 0\end{pmatrix}B_\mu + \frac{X}{4}g'gB_\mu\begin{pmatrix}-2W_{3L}^\mu{\nu_L}^2 & ... \\ ... & 0\end{pmatrix} \\ &+ \frac{X^2}{4}g'^2B_\mu B^\mu\begin{pmatrix}{\nu_L}^2 & 0 \\ 0 & 0\end{pmatrix}\Bigg]\end{aligned} \tag{B.8}$$

this is

$$\begin{aligned}\mathrm{Tr}[(D_\mu\Delta_L)^\dagger(D^\mu\Delta_L)] =& \frac{g^2}{2}\nu_L^2({W_{1L}}^2+{W_{2L}}^2) + g^2\nu_L^2W_{3L}^2 - Xgg'\nu_L^2B_\mu W_{3L}^\mu \\ &+ \frac{X^2}{4}g'^2\nu_L^2B_\mu B^\mu\end{aligned} \tag{B.9}$$

And the third term,

$$\mathrm{Tr}[(D_\mu\Delta_R)^\dagger(D^\mu\Delta_R)] = \mathrm{Tr}\Bigg[\frac{g^2}{4}\begin{pmatrix}({W_{1R}}^2+{W_{2R}}^2){\nu_R}^2+4{\nu_R}^2{W_{3R}}^2 & ... \\ ... & {\nu_R}^2({W_{1R}}^2+{W_{2R}}^2)\end{pmatrix}$$
$$+\frac{X}{4}gg'\begin{pmatrix}-2W^\mu_{3R}{\nu_R}^2 & ... \\ ... & 0\end{pmatrix}B_\mu+\frac{X}{4}g'gB_\mu\begin{pmatrix}-2W^\mu_{3R}{\nu_R}^2 & ... \\ ... & 0\end{pmatrix}$$
$$+\frac{X^2}{4}g'^2B_\mu B^\mu\begin{pmatrix}{\nu_R}^2 & 0 \\ 0 & 0\end{pmatrix}\Bigg] \tag{B.10}$$

$$\mathrm{Tr}[(D_\mu\Delta_R)^\dagger(D^\mu\Delta_R)] = \frac{g^2}{2}\nu_R^2({W_{1R}}^2+{W_{2R}}^2)+g^2\nu_R^2W_{3R}^2-Xgg'\nu_R^2B_\mu W^\mu_{3R}$$
$$+\frac{X^2}{4}g'^2\nu_R^2B_\mu B^\mu \tag{B.11}$$

We can see from the calculations, that the parity of the left and the right is invariant. Left-handed and right-handed gauge fields are symmetric and commute under "L - R" or "R - L". This is also the reason why one can choose either order in Equation (3.32). The $W_{1,2}$ terms and $W^{+,-}$ have an equivalent relation,

$$W^2_{1L,R}+W^2_{2L,R}={W^+_{L,R}}^2+{W^-_{L,R}}^2=2W^+_{\mu L,R}W^{-,\mu}_{L,R} \tag{B.12}$$

Sum all three terms, we have

$$\begin{aligned}
&\mathrm{Tr}\big[(D_\mu\phi)^\dagger(D^\mu\phi)+(D_\mu\Delta_L)^\dagger(D^\mu\Delta_L)+(D_\mu\Delta_R)^\dagger(D^\mu\Delta_R)\big]\\
=\;&\frac{g^2}{2}(\kappa^2+\kappa'^2+2\nu_L^2)W^+_{L,\mu}{W_L^-}^{,\mu}+\frac{g^2}{2}(\kappa^2+\kappa'^2+2\nu_R^2)W^+_{R,\mu}{W_R^-}^{,\mu}\\
&+\frac{g^2}{4}(\kappa^2+\kappa'^2+4\nu_L^2)W_{3L}^2+\frac{g^2}{4}(\kappa^2+\kappa'^2+4\nu_R^2)W_{3R}^2\\
&-\frac{g^2}{2}(\kappa^2+\kappa'^2)W_{3L}W_{3R}-g^2\kappa\kappa'W^-_{L,\mu}W_R^{+,\mu}\\
&-g^2\kappa\kappa'W^-_{R,\mu}W_L^{+,\mu}-Xgg'(\nu_L^2W_{3L,\mu}B_\mu+\nu_R^2W_{3R,\mu}B^\mu)\\
&+\frac{X^2}{4}g'^2(\nu_L^2+\nu_R^2)B_\mu B^\mu
\end{aligned} \tag{B.13}$$

These were taken into use in the main part of the thesis.

C "Integrate Out"

The "integrate out" method is widely used in diagrams or processes that contain large massive mode fields and relatively light modes, where the heavy mode degree of freedoms can be reduced to the mass point as the first order of the expansion series [119]. Mathematically, the "integrate out" procedure indicates to extract out the free terms (terms without the integrating parameter, usually the first term) of the integral after the Taylor expansion. Let us explicitly derive the formulation by considering a toy example with two coupled scalars. The Lagrangian density of the system could be

$$\mathscr{L}_{scalar}(\Phi,\phi) = \partial_\mu\Phi\partial^\mu\Phi + \partial_\mu\phi\partial^\mu\phi + \frac{1}{2}M\Phi^2 + \frac{1}{2}m\phi^2 + \lambda\Phi\phi^2 \tag{C.1}$$

Φ, ϕ are the heavy mode and the light mode, respectively, which we suggest $M \gg m$. The last term $\lambda\Phi\phi^2$ refers to the couple of the two fields, and may indicate the decay process from the heavy mode to the light mode. The dimension of both fields are one power to the mass $[\phi] = [\Phi] = m$, i.e. the coupling λ in the term has the dimension to mass instead of dimensionless. The generating functional of the system is

$$Z[\Phi,\phi] = \int \mathcal{D}\phi\mathcal{D}\Phi \mathrm{e}^{-S[\Phi,\phi]} = \int \mathcal{D}\phi \mathrm{e}^{-S_{eff}[\phi]} \tag{C.2}$$

We are calculating in the 4-dimensional Euclidean space so a minus sign has been taken into account in the exponential through wick rotation $ix^0 \to -x^0$ [19]. We have ignored the normalization factor for simplification. The action in this way is

$$S[\Phi,\phi] = \int d^4x \frac{1}{2}(\partial_\mu\Phi\partial^\mu\Phi + \partial_\mu\phi\partial^\mu\phi + M\Phi^2 + m\phi^2 + \lambda\Phi\phi^2) \tag{C.3}$$

the effective action in the generating functional is obtained by integrating the heavy mode Φ. Now we perform this integral carefully:

$$\int \mathcal{D}\Phi \mathrm{e}^{-S[\Phi,\phi]} = \int \mathcal{D}\Phi \mathrm{e}^{-\int d^4x[\Phi(-\partial_\mu\partial^\mu + M^2)\Phi + \lambda\Phi\phi^2}]\mathrm{e}^{-S_0[\phi]} \tag{C.4}$$

We have used the fact that the integral to the total derivative in the whole area goes to zero because the Lagrangian satisfies the equations of motions, i.e. the Euler-Lagrange equation. Define $G^{-1} \equiv -\partial_\mu\partial^\mu + M^2$, and G is the Green function.

$$(-\partial_\mu\partial^\mu + M^2)G(x-y) = \delta(x-y) \tag{C.5}$$

where the functional integral with respect to Φ is a general Gaussian integral [20]. We will obtain:

$$\int \mathcal{D}\Phi \mathrm{e}^{-S[\Phi,\phi]} = \mathcal{C}\mathrm{e}^{-\int d^4x \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi + m\phi^2 + \frac{1}{2}\lambda^2 G\phi^4)} \tag{C.6}$$

where $\mathcal{C} = (\frac{(2\pi)^4}{\det(G^{-1})})^{-\frac{1}{2}}$ is the factor of the functional Gaussian integral. The solution of the differential equation (C.5) in the momentum space is

$$G(x,y) = \frac{i}{p^2 + M^2} = \frac{1}{M^2}\frac{i}{1+\frac{p^2}{M^2}} \approx \frac{i}{M^2}(1 - \frac{p^2}{M^2} + \frac{p^4}{M^4} + \cdots) \tag{C.7}$$

where we use the assumption $p^2 \ll M^2$ in the heavy mode. Insert (C.7) into (C.6) and (C.2), and only consider the first order.

$$Z[\Phi,\phi] = \int \mathcal{D}\phi \mathrm{e}^{-S_{eff}[\phi]} \approx \mathcal{C}\int \mathcal{D}\phi \mathrm{e}^{-\int d^4x \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi + m\phi^2 + \frac{1}{2}\frac{1}{M^2}\lambda^2\phi^4)} \tag{C.8}$$

In this aspect, the effective action in the generating functional is independent on the heavy mass field Φ. The Φ degree of freedom is contracted into the self-interaction term coupling $\frac{1}{2}\frac{\lambda^2}{M^2}$. The coupling is dimensionless, which makes the Lagrangian purterbatively renormalizable. Thus, the effective field is mathematically reasonable. The effective propagator is widely used in particle physics. In this thesis, the effective action of a fermion-heavy boson interacting model is

$$S_{eff} = -\frac{G_F}{\sqrt{2}}\int d^dx d^dy J^\mu_{L,R} J_{L,R,\mu} \tag{C.9}$$

where $J = \overline{\Psi}^a \mathcal{O}\Psi^a$, and $\mathcal{O}$ are the operators representing vector, axial vector, pseudo-vector, and so on. G_F is the Fermi constant. The diagram in this way reduces to a "4-fermion" self-interact vertex, see Figure C.1.

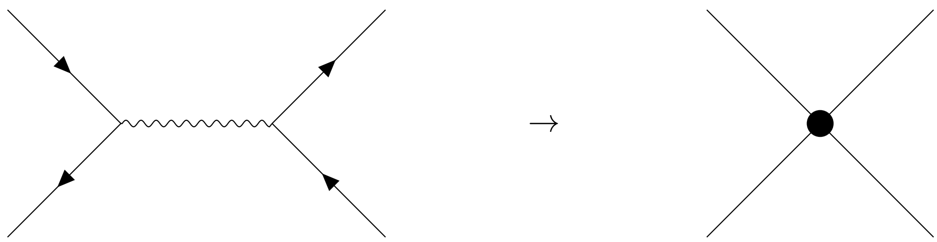


Figure C.1: graphic of "Integrate out" from a fermion-boson full theory to the first order effective theory

Here are only some basic derivations for the effective action. There are papers and lecture notes for discussions on the effective theory, for example see [120, 121]

D Fierz Transformation

Fierz transformation is based on Fierz identity where any bilinear product of two spinors can be rewritten into a linear combination of the products of bilinear spinors with the correspondent coefficient. This rewritten is mean to look for the composition in the spanned space so as to transform the formula and reduce the techniques calculation. Instructively, Fierz identity allows one to change the spinors position in a Lagrangian or an amplitude, or some other physical product by choosing the combinations and adding the coefficient. Fierz identity was first introduced by Markus Fierz in his paper for discussing beta decay process [122]. Readers can find the details in several lecture notes or articles [123, 124, 125]. We mainly follow [123] in our thesis to show a brief derivation.

The products of two spinors are categorised by various of hadronic and leptonic currents with different operators. These currents are choosing physically so that they are Lorentz-invariant bilinear covariant. Define a general product of two spinors as

$$e_I^a(12) \equiv \overline{\psi}_1 \Gamma_I^a \psi_2 \tag{D.1}$$

Γ^a is the operator that determines the type of the product. The subscript I stands for the type. The conventionally used types are (first words in each line stands for the product type)

$$\begin{aligned} \text{scalar} :& \Gamma_S^1 \equiv \mathbb{1} \\ \text{vector} :& \Gamma_V^{1-4} \equiv \gamma^\mu \\ \text{tensor} :& \Gamma_T^{1-6} \equiv \sigma^{\mu\nu}/\sqrt{2} \;\; (\mu < \nu) \\ \text{axial vector} :& \Gamma_A^{1-4} \equiv i\gamma^\mu\gamma^5 \\ \text{pseudo-scalar} :& \Gamma_P^1 \equiv \gamma^5 \end{aligned} \tag{D.2}$$

where γ^μ, γ^5 are gamma matrices and gamma-5 matrix respectively. And $\sigma^{\mu\nu}$ is

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] \tag{D.3}$$

where the superscripts in each operator are Lorentz indices. For example, Γ^{1-4} gives all the operator with the count from 1 to 4, i.e. $\Gamma^1, \Gamma^2, \Gamma^3, \Gamma^4$. From the definitions of five types (D.1), we now define the simplest quadrilinear products

$$e_I(1234) \equiv n_I^2 e_I^a(12) e_{Ia}(34) \tag{D.4}$$

where the coefficients are

$$n_I = \begin{cases} 1 & I = S, V, P \\ -i & I = A \\ \sqrt{2} & I = T \end{cases} \tag{D.5}$$

With the coefficients we have (D.4)

$$\begin{aligned} e_S(1234) &= (\overline{\psi}_1\psi_2)(\overline{\psi}_3\psi_4) \\ e_V(1234) &= (\overline{\psi}_1\gamma^\mu\psi_2)(\overline{\psi}_3\gamma_\mu\psi_4) \\ e_T(1234) &= (\overline{\psi}_1\sigma^{\mu\nu}\psi_2)(\overline{\psi}_3\sigma_{\mu\nu}\psi_4) \\ e_A(1234) &= (\overline{\psi}_1\gamma^\mu\gamma_5\psi_2)(\overline{\psi}_3\gamma_\mu\gamma_5\psi_4) \\ e_P(1234) &= (\overline{\psi}_1\gamma_5\psi_2)(\overline{\psi}_3\gamma_5\psi_4) \end{aligned} \tag{D.6}$$

Now we are going to use Fierz identities to transform the quadrilinears into another one that contains different order of the spinors in the bilinear products, i.e. transform the basis of spinors by considering all the linear expansions of operators in the spanned space. The Fierz transformation of the spinors basis order is expressed as

$$e_I(1234) = \sum_J F_{IJ} e_J(1432) \tag{D.7}$$

where F_{IJ} are numerical coefficients, and I, J run over S, V, T, A, P. The Fierz transformation behaves equivalent as basis transformation in matrix algebra where $e_I(1234)$ can be written as the linear combinations of $e_J(1432)$. In order to gain the coefficient, we first add the matrix indices

$$n_I^2 e_I(12)_{ab} e_I(34)_{cd} = n_J^2 \sum_J F_{IJ} e_I(14)_{ad} e_J(32)_{cb} \tag{D.8}$$

then contract with $e_K(41)_{da} e_K(23)_{bc}$ and relabel the index $K \to J$, we will obtain

$$\begin{aligned} F_{IJ} =& \frac{n_I^2}{n_J^2}\frac{1}{16}\mathrm{Tr}[\Gamma_I^a \Gamma_J^b \Gamma_{Ia} \Gamma_{Jb}] \\ =& \frac{n_I^2}{4n_J^2} f_{IJ} \end{aligned} \tag{D.9}$$

where we use the properties of gamma matrices and introduce f_{IJ}

$$\begin{aligned} \{\gamma^\mu, \gamma^\nu\} &= 2g^{\mu\nu}\mathbb{I}_{4\times 4} \\ \mathrm{Tr}[\Gamma_I^a \Gamma_{Jb}] &= 4\delta_{IJ}\delta_b^a \\ \Gamma_I^a \Gamma_J^b \Gamma_{Ia} &= f_{IJ}\Gamma_J^b \end{aligned} \tag{D.10}$$

Finally, combine Equation (D.9) and (D.10), we have the coefficients in the matrix form

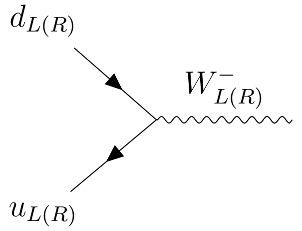
$$\mathbf{F}_{IJ} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{8} & -\frac{1}{4} & \frac{1}{4} \\ 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & -1 \\ 3 & 0 & -\frac{1}{2} & 0 & 3 \\ -1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 1 \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{8} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} \tag{D.11}$$

For instance, the scalar quadrilinear with spinor order (1234) can be rewritten in the new order (1432) as

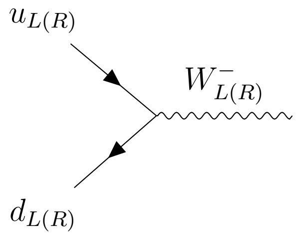
$$e_S(1234) = \frac{1}{4}[e_S(1432) + e_V(1432) + \frac{1}{2}e_T(1432)) - e_A(1432) + e_P(1432)] \tag{D.12}$$

E Feynman Rules

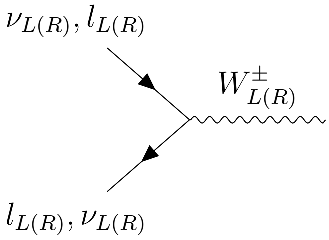
Here is the vertices we use in this thesis:



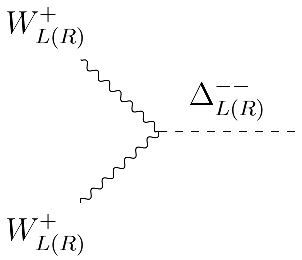
$$-\frac{i}{\sqrt{2}}\gamma_\mu g_{L(R)} P_{L(R)} V^*_{ud} \tag{E.1}$$



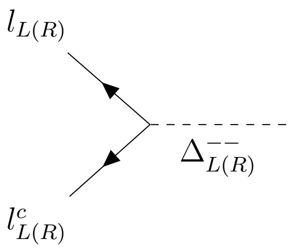
$$-\frac{i}{\sqrt{2}}\gamma_\mu g_{L(R)} P_{L(R)} V^{ud} \tag{E.2}$$



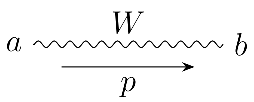
$$-\frac{i}{\sqrt{2}}\gamma_\mu g_{L(R)} P_{L(R)} \tag{E.3}$$



$$-\frac{i}{\sqrt{2}} g^2_{L(R)} \nu_{L(R)} \tag{E.4}$$



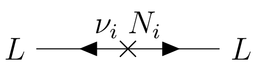
$$-\frac{i}{\sqrt{2}\nu_R} M^{diag}_\nu P_{L(R)} \tag{E.5}$$



$$\frac{ig^{ab} - (1-\epsilon)\frac{p^a p^b}{p^2}}{p^2 - m_W^2 - i\epsilon} \tag{E.6}$$

where usually $\epsilon = 0$ is the Landau gauge and $\epsilon = 1$ is the Feynman gauge. One can choose any gauge, but the physics of the Lagrangian, e.g. EoMs (equations of motions) will stay unchanged. We will use Feynman gauge in our calculations.

The propagator of the Majorana neutrino is not the same as the Dirac one's. We give the propagator and the calculation below. L, R refer to the chirality of the vertices sandwiching the propagator and $P_{L,R}$ are the correspondent projection operators. C is the charge conjugate operator. ν, N give the light or heavy neutrinos.



$$P_L \frac{-i(\not{p} + m_{\nu,N})}{p^2 - m_{\nu,N}^2 + i\epsilon} P_L C \qquad \text{(E.7)}$$

Now we derive the neutrino operator from the two point correlation function in the position basis [103]. Before doing it, let us first derive the Majorana spinors completeness relation. From the property of the Majorana field we have

$$\begin{aligned}
\Psi &= C\overline{\Psi}^T \\
&= C[(ve^{ipx})^\dagger \gamma^0] \\
&= C[\overline{v}e^{-ipx}]^T \\
&= C\overline{v}^T e^{-ipx} \\
&= ue^{-ipx}
\end{aligned} \qquad \text{(E.8)}$$

then this implies,

$$u = C\overline{v}^T \qquad \text{(E.9)}$$

also in another way around, we have

$$v = C\overline{u}^T \qquad \text{(E.10)}$$

then the completeness relation goes

$$\begin{aligned}
\sum_{s,s'} u(p,s)v^T(p,s') &= \sum_{s,s'} u(p,s)(C\overline{u}^T(p,s'))^T \\
&= \sum_{s,s'} u(p,s)\overline{u}(p,\prime)C^T \\
&= -(\not{p} + m)C
\end{aligned} \qquad \text{(E.11)}$$

where we use normal fermion completeness relation $\sum u(p,s)\overline{u}(p,s') = \not{p} + m$ and the property $C^T = -C$. Explicitly, the sum over calculation concludes the transpose of the spinor since u and v are vector like quantities. We can use the same rules to obtain the rest relations. Combining with those Dirac spinors', the relations are

$$\sum_{s,s'} \overline{u}^T(p,s)\overline{v}(p,s') = C^{-1}(\not{p}-M) \qquad \sum_{s,s'} \overline{v}^T(p,s)\overline{u}(p,s') = C^{-1}(\not{p}+M)$$
$$\sum_{s,s'} v(p,s)u^T(p,s') = -(\not{p}-m)C$$
$$\sum_{s} \overline{u^s}(p,s)u^s(p,s) = \not{p}+m \qquad \sum_{s} \overline{v^s}(p,s)v^s(p,s) = \not{p}-m \tag{E.12}$$

The two point Green function goes

$$\begin{aligned}
&\langle 0| T\big(\nu_{lL}(x_1)\nu_{l'L}(x_2)\big) |0\rangle \\
=& \langle 0| T \int \frac{d^3p_1}{(2\pi)^3}\frac{1}{2p_1^0}\sum_s \left[a_s(p_1)u_{lL}(p_1,s)\mathrm{e}^{-ip_1\cdot x_1} + a_s^\dagger(p_1)v_{lL}(p_1,s)\mathrm{e}^{ip_1\cdot x_1}\right] \\
&\cdot \int \frac{d^3p_2}{(2\pi)^3}\frac{1}{2p_2^0}\sum_s \left[a_s(p_2)u_{l'L}(p_2,s)\mathrm{e}^{-ip_2\cdot x_2} + a_s^\dagger(p_2)v_{l'L}(p_2,s)\mathrm{e}^{ip_2\cdot x_2}\right] |0\rangle \\
=& \int \frac{d^3p_1}{(2\pi)^3}\frac{1}{2p_1^0}\int \frac{d^3p_2}{(2\pi)^3}\frac{1}{2p_2^0}\mathrm{e}^{-ip_1\cdot x_1}\mathrm{e}^{ip_2\cdot x_2}\sum_{s,s'} u_{lL}(p_1,s)v_{l'L}(p_2,s') \\
&\cdot \langle 0| a_s(p_1)a_{s'}^\dagger(p_2) |0\rangle + \big(1\leftrightarrow 2, l\leftrightarrow l'\big) \\
=& \int \frac{d^3p_1}{(2\pi)^3}\frac{1}{2p_1^0}\mathrm{e}^{-ip_1\cdot x_1}\int \frac{d^3p_2}{(2\pi)^3}\frac{1}{2p_2^0}\mathrm{e}^{ip_2\cdot x_2}\sum_{s,s'} u_{lL}(p_1,s)v_{l'L}(p_2,s')(2\pi)^3 2p_1^0\delta^3(\vec{p_1}-\vec{p_2}) \\
&+ \big(1\leftrightarrow 2, l\leftrightarrow l'\big) \\
=& \int \frac{d^3p_1}{(2\pi)^3}\frac{1}{2p_1^0}\mathrm{e}^{-ip_1\cdot(x_1-x_2)}\sum_{s,s'} u_{lL}(p_1,s)v_{l'L}(p_1,s') + \big(1\leftrightarrow 2, l\leftrightarrow l'\big)
\end{aligned} \tag{E.13}$$

where we use the general quantized wave function of fermions,

$$\nu_a(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2p^0}\sum_s \left[a_s(p)u_a(p,s)\mathrm{e}^{-ip\cdot x} + a_s^\dagger(p)v_a(p,s)\mathrm{e}^{ip\cdot x}\right] \tag{E.14}$$

and the commutation relation,

$$[a_s(p_1), a_s^\dagger(p_2)] = (2\pi)^3 2p_1^0\delta^3_{ss'}(\vec{p}-\vec{p}') \tag{E.15}$$

and we also take $a|0\rangle = \langle 0| a^\dagger = 0$. Inserting the completeness relation back into the propagator we obtain,

$$
\begin{aligned}
&\langle 0|\, T\big(\nu_{lL}(x_1)\nu_{l'L}(x_2)\big)\, |0\rangle \\
=&\int \frac{d^3p_1}{(2\pi)^3}\frac{1}{2p_1^0} P_L\big[-(\not{p}_1+m_\nu)P_L C \mathrm{e}^{-ip_1\cdot(x_1-x_2)} - (-\not{p}_1+m_\nu)P_L C \mathrm{e}^{ip_1\cdot(x_1-x_2)}\big] \\
=&\int \frac{d^4p_1}{(2\pi)^4}\mathrm{e}^{-ip_1\cdot(x_1-x_2)} P_L \frac{-i(\not{p}_1+m_\nu)}{p_1^2-m_\nu^2+i\epsilon} P_L C \qquad \text{(E.16)}
\end{aligned}
$$

This first-ordered propagator is considered as two in-going neutrino from the electron-boson-neutrino vertex that “meet” each other to propagate. This is only reasonable when we are dealing with the Majorana particles. For other cases, e.g. two out-going $\overline{\nu\nu}$ or one in-going with one out-going $\nu\overline{\nu}$, the C operator is rearranged in the equation, and the normal fermion part of the propagator remains the same [103]. The neutrino mass m_ν is in fact the effective mean mass $\langle m_\nu\rangle = \sum_i U_{ei}^2 m_{\nu i}$. The out-going neutrinos propagator $\langle 0|\, T\big(\nu_{lL}(x_1)\nu_{l'L}(x_2)\big)\, |0\rangle$ is

$$
P_L \frac{-iC^{-1}(\not{p}+m_{\nu,N})}{p^2-m_{\nu,N}^2+i\epsilon} P_L \qquad \text{(E.17)}
$$

Equation (E.17) and (E.7) are the propagators in the momentum space. Again to emphasise, the projection operators come from the vertices in two ending sides of the propagator. We use the left-handed neutrinos propagator as an example for the calculation, generally, one does not need to consider about the chirality in the pure propagator derivation. However, the chirality reduces the expression of the propagator. Take (E.17) as an example.

$$
\begin{aligned}
&P_L \frac{-i(\not{p}+m_{\nu,N})}{p^2-m_{\nu,N}^2+i\epsilon} P_L C = \frac{1-\gamma_5}{2}\frac{-i(\not{p}+m_{\nu,N})}{p^2-m_{\nu,N}^2+i\epsilon}\frac{1-\gamma_5}{2} C \\
=&\frac{-i\not{p}(\frac{1+\gamma_5}{2}) + m_{\nu,N}(\frac{1-\gamma_5}{2})}{p^2-m_{\nu,N}^2+i\epsilon}\frac{1-\gamma_5}{2} C \\
=&\frac{m_{\nu,N}}{p^2-m_{\nu,N}^2+i\epsilon} P_L C \qquad \text{(E.18)}
\end{aligned}
$$

where we the anticommutative relation of Dirac matrices $\{\gamma^5,\gamma_\mu\} = \gamma_5\gamma_\mu + \gamma_\mu\gamma_5 = 0$ and $(\gamma_5)^2 = \mathbf{1}$.

F Trace Technology

Trace technology is to calculate the completeness summation of spinors. It is the gamma matrices and its combinations. One can find the mathematical interpretation and the detailed calculation in text books or lecture notes, for example, see chapter 5 of [19]. There are properties we consider in the trace technology ($\mathbf{A}, \mathbf{B} \dots$ are arbitrary square matrices):

$$\begin{aligned} &\mathrm{Tr}[\mathbf{A}+\mathbf{B}] = \mathrm{Tr}[\mathbf{A}] + \mathrm{Tr}[\mathbf{B}] \ \ \text{(linearity)} \\ &\mathrm{Tr}[\mathbf{A}] = \mathrm{Tr}[\mathbf{A}^T] \\ &\mathrm{Tr}[\mathbf{ABC}] = \mathrm{Tr}[\mathbf{BCA}] = \mathrm{Tr}[\mathbf{CAB}] \ \ \text{(cyclicity)} \end{aligned} \tag{F.1}$$

The cyclic property works for any number of matrices. Here are traces for gamma matrices in 4-dimensional Minkowski space.

$$\begin{aligned} &\mathrm{Tr}[\mathbf{1}] = 4 \\ &\mathrm{Tr}[\gamma^\mu] = 0 \\ &\mathrm{Tr}[\text{any odd umber of}\gamma^\mu] = 0 \\ &\mathrm{Tr}[\gamma^\mu\gamma^\nu] = 4g^{\mu\nu} \\ &\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) \\ &\mathrm{Tr}[\gamma_5] = 0 \\ &\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma_5] = 0 \\ &\mathrm{Tr}[\text{any odd number of}\gamma^\mu\gamma_5] = 0 \\ &\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5] = -4i\epsilon^{\mu\nu\rho\sigma} \\ &\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma_\mu] = \mathrm{Tr}[-2\gamma^\nu] = 0 \\ &\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu] = \mathrm{Tr}[4g^{\nu\rho}] = 8 \\ &\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu] = \mathrm{Tr}[-2\gamma^\sigma\gamma^\rho\gamma^\nu] = 0 \end{aligned} \tag{F.2}$$

where $\epsilon^{\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma}$ is the Levi-Civita tensor. We also use the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{I}_{4\times 4} \tag{F.3}$$

The trace of the leptonic part in LL (4.44):

$$
\begin{aligned}
&\sum_{s,s'} \overline{v}_c^{s'}(p_2)(1-\gamma^5)_{cd} u_d^s(p_1) \overline{u^s}_a(p_1)(1+\gamma^5)_{ab} v_b^{s'}(p_2) \\
=&(\not{p}_2 - m_e)_{bc}(1-\gamma^5)_{cd}(\not{p}_1 + m_e)_{da}(1+\gamma^5)_{ab} \\
=&\frac{1}{4}\mathrm{Tr}[(\not{p}_2 - m_e)(1-\gamma^5)(\not{p}_1 + m_e)(1+\gamma^5)] \\
=&2\frac{1}{4}\mathrm{Tr}[\not{p}_1\not{p}_2] \\
=&2(p_1 \cdot p_2) \\
=&2(E_1 E_2 - p_1 p_2 \cos\xi)
\end{aligned}
\tag{F.4}
$$

G Lists

G.1 List of Figures

G.2 List of Tables

H Bibliography

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Danksagung

In the end, I would like to use a small paragraph to say "thank you". I am extremely grateful to all those who gave advice and guidance to me, to all those who have accompanied me, encouraged me, and supported me during my study in Master in Universität Heidelberg.
First, it is my honor to study with my supervisor Prof. Dr. Dr. h.c. Manfred Lindner who always gives me instruction and guidance on my learning and researches, and at the same time, I appreciate having the opportunity to complete my thesis in the Max-Planck-Institut für Kernphysik. Many thanks to Dr. Lukáš Gráf, Oliver Scholer, who offers me advice and hints on researches as well as on the master thesis. Many thanks to Aika Tada, Robert Hammann, Jan Gräfje, always being friendly and nice, whom I worked with in the same office at the MPIK. Many thanks to my friends and roommates, Ruoyu Xing, Datong Qiu, Bailin He, Yechao Han, who are always kind in daily life. And to Dini Li, thanks for accompanying me in the whole time.
Finally, I would like to express special thanks to my family, especially to my parents.

Erklärung:

Ich versichere, dass ich diese Arbeit selbstständig verfasst habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Heidelberg, den (Datum) .... Kefei Zou

30.05.2022

邹可飞